Colored Triple Linking Number

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Introduction

In an important work, Cimasoni and Florens [3, 4] introduced the notion of a colored link (literally a link whose components have been grouped together by colors). These allow one to define new link invariants by treating components of a single color like they are a single component. In this project, we consider the so called triple linking number [9]. This invariant can be defined in terms of the intersections of a collections of surfaces bounded by that link. Our new invariant can be used to prove that certain colored links are not boundary links - meaning that any surfaces bounded by their colored sublinks must intersect each other.

Definitions

A *link* is a union of embedded circles in 3-dimensional space. An n-colored link is a link whose components have been colored with integers 1 through n. We denote the i-colored sublink of L by L_i.
A C-complex for an n colored link is a union of surfaces $F = F_1 \cup \cdots \cup F_n$ where each F_i is an embedded surface bounded by L_i. These surfaces are permitted to intersect each other in clasps.

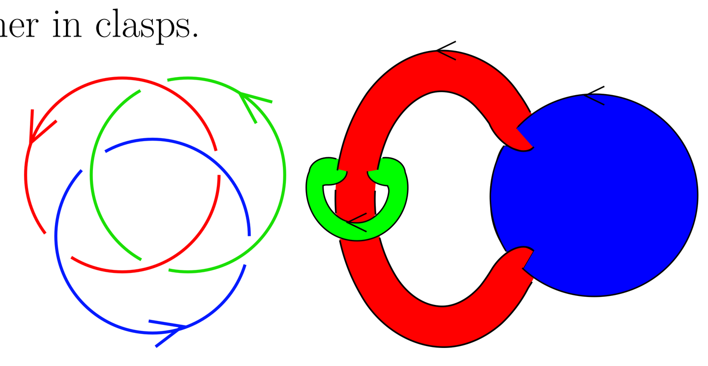

Figure 1:Colored Borromean rings and a C-complex.

The Classical Triple Linking Number

In the 1950's, Milnor defined a collection of high order linking invariants (tools that detect increasingly subtle interactions between the components of a link). Even the first of these, the so-called triple linking number, has been of intense study ever since.
The triple linking number via a C-complex Let L be a link (whose every component is a different color). Mellor-Melvin [8] give a formula for the triple linking number in terms of a C-complex F. We recall it now. As we present their algorithm, we follow it for the example of Figure 1. Here the red component will be numbered 1, the blue component numbered 2, and the green component numbered 3.
For each component L_k of the link, pick a base point, p_k and then follow the link a write down a word that records an x_i^{+1} (or x_i^{-1}) whenever L_k crosses F_i in the positive (or negative) direction. Call this word a *claspword* and denote it $w_k(F)$. Following L_1 starting at the arrow, we see, in order: a negative clasp with F_3, a positive clasp with F_2, a positive clasp with F_3, and a negative clasp with F_2. We encode this in a claspword $w_1(F) = x_3^{-1}x_2x_3x_2^{-1}$
Computing all three clasp words:

$$w_1(F) = x_3^{-1}x_2x_3x_2^{-1},\ w_2(F) = x_1^{-1}x_1,\ w_3(F) = x_1^{-1}x_1.$$

For any such word w, the *Magnus coefficient* of w, $e_{ij}(w)$ records with sign how many times an x_i occurs before an x_j:

$$e_{23}(w_1) = 1,\ e_{31}(w_2) = 0,\ e_{12}(w_3) = 0,$$

Finally, the triple linking number is a sum of these:

$$\mu_{123}(L) = e_{12}(w_3(F)) + e_{23}(w_1(F)) + e_{31}(w_2(F)) = 1.$$

Modulo the linking numbers of L, it is independent of the choice of C-complex and so gives an invariant of L. This combinatorial object has been used to prove that links do not admit disjoint embedded surfaces, to classify when links admit equivalent C-complexes, and to bound below the number of clasps in a C-complex. [2, 6, 7],

Triple Linking Number for a Colored Link

Our goal is to extend the notion of triple linking number to the setting colored links. For example, the link bounding the C-complex below.

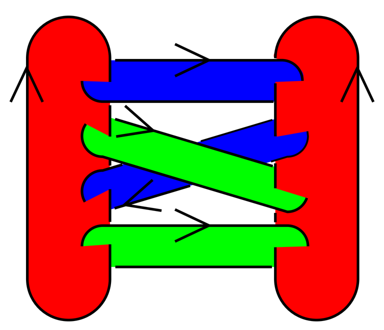

Figure 2:A six component, 3-colored link along with a C-complex.

As a first step notice that each component of a link produces a clasp-word just as in the classical setting. If there are multiple components with the same color, then there will be multiple claspwords associated with that color. For technical reasons, we will assume that the linking numbers (the number of positive clasps minus the number of negative clasps seen by any one component) is zero.

$$\begin{aligned} w_1^1(F) &= x_2x_2^{-1}, & w_1^2(F) &= x_2^{-1}x_2 \\ w_2^1(F) &= x_3x_1x_3^{-1}x_1^{-1}, & w_2^2(F) &= x_3^{-1}x_3x_1x_1^{-1} \\ w_3^1(F) &= x_2^{-1}x_2, & w_3^2(F) &= x_2^{-1}x_2 \end{aligned}$$

For each clasp word we can compute its Magnus coefficient.
$e_{23}(w_1^1) = 0, e_{23}(w_1^2) = 0,\ e_{31}(w_2^1) = 1, e_{31}(w_2^1) = 0,\ e_{12}(w_3^1) = 0, e_{12}(w_3^2) = 0$
Next we add together all of the Magnus coefficients with the same color.

$$e_{23}(F_1) = 0,\ e_{31}(F_2) = 1,\ e_{12}(F_3) = 0$$

Finally, $\mu_{123}(L) = e_{12}(w_3(F)) + e_{23}(w_1(F)) + e_{31}(w_2(F))$

$$\mu_{123}(L) = 1$$

Our key result is that this quantity is independent of the C-complex bounded by L and so is an invariant of L.

Theorem

If the link L has zero linking numbers, then the colored triple linking number $\mu_{ijk}(L)$ is independent of the C-complex and so is an invariant of the link L.
If L does not have vanishing linking number, then $\mu_{ijk}(L)$ is well defined modulo the linking numbers of L.

What can this do? Since this computation is independent of the choice of C-complex, the colored triple linking number computed using one C-complex bounded by L tells us something about *every* C-complex bounded by L. For instance, every C-complex bounded by the link of Figure 2 must result in a triple linking number of -1. In particular, each such C-complex must have some clasps.

Why is this Well Defined?

By [3, 5] any two C-complexes for a fixed link are related by a sequence of geometric moves:

- Ambient isotopy (bending and twisting the C-complex but not changing its underlying shape)
- Adding a tube from a component to itself. (No claspwords will change.)
- The (T2), (T3), and (T4) moves. (Pictured below)

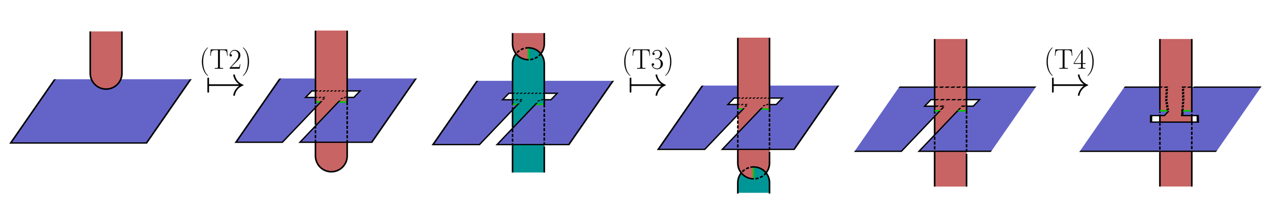

Figure 3:The (T2), (T3), and (T4) moves. respectively.

By checking the effect of each of these moves, we prove the colored triple linking number is independent of the choice of C-complex.

Changing the Base Point: Why We Need Zero Linking Numbers.

Our definition also relies on a choice of basepoint (the arrow at which we start reading off out claspwords). We need to check that the triple linking number is unchanged by moving the base point.

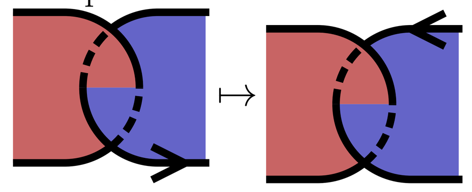

Figure 4:Moving a base point (the arrow) through a clasp.

To do so, we determine the effect of moving the basepoint over a clasp (Figure 4) on the colored triple linking number. We compute the triple linking number before and after the move. The only clasp word that changes is w_3, coming from the blue component. Before changing the base point $w_3 = x_1 \cdot \gamma$ where γ is some word. After moving the base point, $w_3 = \gamma \cdot x_1$. Before the base point is moved, x_1 is counted exactly once for each x_2 in γ, which is precisely the linking number. When the base point is moved, x_1 will not be counted at all, hence it changes exactly by the linking number. Thus, as long as the linking number is 0, the triple linking number remains the same regardless of the base point selected.

Future Directions

- When the linking numbers do not vanish the (classical or colored) triple linking number is not well defined as an integer. The precise indeterminacy of the classical triple linking number has been computed [1]. Can the same be done for the colored triple linking number?
- The classical triple linking number completely determines when two links admit homeomorphic C-complexes (informally, C-complexes which are the same shape but with the 3-dimensional space differently). [7]. Does the colored triple linking number do the same for colored links?

Acknowledgments

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