Renormalization Group Method on Reaction-Diffusion Partial Differential Equations and Solutions to the Resulting Complex Ginzburg-Landau Equation

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In order to find solutions to a set of reaction-diffusion PDEs, it will be shown that specific kinds of PDEs can immediately be reduced to a form of the complex Ginzburg- Landau equation (CGLE). Using the renomalization groups method, it is proven that given some reaction-diffusion equations, this is exactly the case. After some short matrix calculations, any reaction-diffusion equation of similar form can be reduced to an equation with known wave solutions. A wave solution for the specific set of PDEs in this paper is found, along with discussion on the stability of the solution.

Introduction

Perturbation theory is a branch of mathematics that allows for useful information to be obtained from perturbed relationships, or to create approximate solutions to problems that cannot be solved analytically. An example of a perturbation problem could look like $x^2 - x + \varepsilon = 0$ where $0 < \varepsilon << 1$. Typically, a solution would begin with solving the unperturbed problem $x^2 - x = 0$. Then the solution of the perturbed problem can follow by expanding x as $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + ...$ For algebraic expressions, perturbation methods are typically unneeded as the expressions either have better methods of approximation or can be solved analytically.

Mathematics has moved far beyond simple algebraic expressions, especially for describing natural phenomenon. The most common systems for modeling complex behaviors are systems of partial differential equations (PDEs). There are endless forms that systems of PDEs could take, however, within these forms, the PDEs can be classified by their behavior. By classifying PDEs by their behavior, it allows for more meaningful results to be produced. Reaction- diffusion equations are one such type of classification and are often solved with perturbation methods. PDEs that display reaction-diffusion behavior often have points of stability that cause points of instability, along with points of instability that cause points of stability, which are overall called bifurcation points. Bifurcation points are essential in finding solutions of some PDEs.

Perturbation theory by itself is powerful, but can be enhanced by combining it with different methods. One method that utilizes the power of perturbation methods is the renormalization groups method (RG). Despite the effectiveness of this method, there are some foundational topics that are yet to be researched, but there will be no attempt to address these foundational concepts in this paper, as the reason that renormalization groups are effective is not necessary for work in the field of reaction-diffusion systems.

The paper is organized as follows. In section II sources that describe the RG method and reaction-diffusion are summarized as background information. In section III a generalized solution to a set of reaction-diffusion PDEs is obtained using the RG method with envelops, along with solutions to the general equation. Then, in part IV the results and application are summarized.

Background Information

RG methods were first applied to partial differential equations without a stochastic element in the paper *Intermediate Asymptotics and Renormalization Group Theory* by Goldenfeld et al. [5]. It was in this paper that the foundation for RG methods in reaction-diffusion equa- tions began. Goldenfeld et al released another paper titled *Anomalous Dimensions and the Renormalization Group in a Nonlinear Diffusion Process* which greatly expanded the use of perturbation methods into not only the field of PDEs, but nonlinear PDEs with diffusion pro- cesses [6]. Shortly after Goldenfeld's papers were published, Teiji Kunihiro published a paper on the geometric meaning of RG methods and laid some foundational work on the method, giving it rigor that it had previously been lacking, specifically in the geometric meaning of the method [10]. In addition to giving the geometric meaning, Kunihiro produced a more repeat- able method of solving problems with the RG method. Kunihiro also generalized the method to systems of PDEs, making it a very useful method for finding relative solutions to systems of PDEs, which will be done in this paper.

It was well known at the time that methods involving asymptotic analysis produced am- plitude equations. In the paper titled *Renormalization Group Theory for Global Asymptotic Analysis* by Chen et al. proved that the amplitude equations that arise from other meth- ods of asymptotic analysis, such as the method of multiple scales, are in fact the same result determined from renormalization groups [4]. Continuing their work on this Chen et

al. pro- duced another paper titled *Renormalization group and singular perturbations: Multiple scales, boundary layers, and reductive perturbation theory* where they again showed that RG methods produce the same results as other asymptotic methods [3]. In addition to this, they showed that RG methods are much more general than other methods, as the method doesn't require any assumptions about the structure of the system, while in some cases even having superior accuracy to the other asymptotic methods. Additionally, they also used the method to describe spatial systems near bifurcation points [3].

The ability to pull useful information about a system near a bifurcation point cannot be overstated. However, the accuracy of solutions with this method quickly dwindles beyond the point being considered as the center. This means that anything outside of a very small neighborhood around a bifurcation point would no longer be representative of the system. In order to make progress on non-Markovian systems, systems which have short term memory and require more stable solutions, Torabi and Davidsen in their paper titled *Pattern forma- tion in reaction-diffusion systems in the presence of non-Markovian diffusion* used reductive perturbation methods to analyze spatial systems with memory [7].

When mathematicians work in spatial systems it is often useful to express the system in the form of an integral and express related constants with RG equations. Alternatively, in systems with partial differential equations, using analytic methods in tandem with perturbation methods can also result in integrals with various constants being determined by RG equations, this alternative method is what Ei et al. did in their paper titled *Renormalization-Group Method for Reduction of Evolution Equations; Invariant Manifolds and Envelopes*. In this paper the time t0 = t is justified, resulting in a Markovian system. The idea of setting t0 = t is not a new concept, but the justification previously was that it produced useful results. Now that there is justification for that parameter, expanding the system outside of simple Markovian systems would be a much easier task. This is exactly what Torabi and Davidsen did in their paper. They began with this equation:

$$\frac{\partial P_\xi(t)}{\partial t} = \int_0^t dt \quad {}_\mu[W_{\xi\mu}(t-t)P_\mu(t) - W_{\mu\xi}(t-t)P_\xi(t)]$$

which is the generalized master equation for a diffusion equation [7]. This equation is time dependent, and the integral, which integrates over all time, makes it difficult to justify reducing this to a Markovian expression. So instead of doing that, Torabi and Davidsen manipulated the equation into a new form with "memory kernels" which looks like this:

$$\frac{\partial P(x,t)}{\partial t} = \int_0^t dt\, K(t-t) D \frac{\partial^2 P(x,t)}{\partial x^2}$$

which is an equation that has been previously derived called the Fokker-Planck equation. This equation can then be split into two separate equations, which can then be treated as a system of PDEs and solved using similar methods to find solutions near bifurcation points [7].

Derivation and Solution to the Complex Ginzburg- Landau Equation

[Note: Due to limitations with our publication software's ability to properly display complex methematical characters, this section will consist of the originally submitted manuscript. pp 81-96]

The following set of reaction-diffusion PDEs, where Δ represents the laplacian, is where the method will begin:

$$\partial_t x = x - \beta x^2 - xy + D_x \Delta x \tag{1}$$

$$\partial_t y = -\alpha y + \beta x^2 + xy + D_y \Delta y \tag{2}$$

These equations by themselves are fine, but it is often useful to shift these, so, letting $(x_0, y_0) = \left(\frac{\alpha}{\alpha\beta+1}, \frac{1}{\alpha\beta+1}\right)$ and defining ζ, η by $x = x_0 + \zeta$, $y = y_0 + \eta$. The PDEs become:

$$\partial_t \zeta = -\frac{\alpha\beta\zeta}{\alpha\beta+1} - \frac{\alpha\eta}{\alpha\beta+1} - \beta\zeta^2 - \zeta\eta + D_x \Delta\zeta \tag{3}$$

$$\partial_t \eta = \frac{(2\alpha\beta+1)\zeta}{\alpha\beta+1} - \frac{\alpha^2\beta\eta}{\alpha\beta+1} + \beta\zeta^2 + \zeta\eta + D_y \Delta\eta \tag{4}$$

Now, given these equations, it is important to manipulate them into the following form:

$$\frac{\partial \mathbf{X}}{\partial t} = H(\mathbf{X}) + \mathbf{D}\Delta\mathbf{X} \tag{5}$$

which can be done easily by letting $\mathbf{D}$ be a diagonalized matrix, with entries D_x and D_y. Then, it can be simply said that

$$H(\mathbf{X}) = \begin{pmatrix} -\frac{\alpha\beta}{\alpha\beta+1}\zeta - \frac{\alpha}{\alpha\beta+1}\eta - \beta\zeta^2 - \zeta\eta \\ \frac{2\alpha\beta+1}{\alpha\beta+1}\zeta - \frac{\alpha^2\beta}{\alpha\beta+1}\eta + \beta\zeta^2 + \zeta\eta \end{pmatrix} \tag{6}$$

Again, the solution will be shifted by some amount, so, let $\mathbf{X_0}$ be a steady solution, then, substituting $\mathbf{X_0} + \mathbf{u}$ into the equation results in $\frac{\partial \mathbf{u}}{\partial t} = H(\mathbf{u}) + \mathbf{D}\Delta\mathbf{u}$. Since X_0 is a steady solution, all of the terms related to it will become 0, leaving only the shifted variable's terms. Then, to approximate the functions, $H(\mathbf{u})$ is expanded with a Taylor series, and coefficients L, M, and N are produced. This results in the equation:

$$\frac{\partial \mathbf{u}}{\partial t} = \widehat{L}\mathbf{u} + M\mathbf{uu} + N\mathbf{uuu} \tag{7}$$

where $\widehat{L} = L + \mathbf{D}\Delta$. Now $\mathbf{u}$, $\widehat{L}$, M, and N will be expanded using perturbation where $\varepsilon = \sqrt{|\mu|}$ and $\chi = sgn(\mu)$.

$$\mathbf{u} = \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ... \tag{8}$$

$$\widehat{L} = \widehat{L}_0 + \mu \widehat{L}_1 + ... \tag{9}$$

$$M = M_0 + \mu M_1 + ... \tag{10}$$

$$N = N_0 + \mu N_0 + ... \tag{11}$$

Plugging equations 8,9, 10, and 11 into equation 7, results in the following:

$$\begin{aligned} \frac{\partial}{\partial t}(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ...) = \\ (\widehat{L}_0 + \mu \widehat{L}_1 + ...)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ...) + \\ (M_0 + \mu M_1 + ...)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ...)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ...) + \\ (N_0 + \mu N_0 + ...)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ...)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ...)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + ...) \end{aligned}$$

Because these are infinite expressions, the correction added by each term diminishes as the number of terms increase. This means that expanding out to ε^3 will produce a model with little loss in accuracy, meaning that every term of ε^4 and greater can be treated as zero, and thus vanish from the resulting equation. Simplifying in the manner reduces the expression to:

$$\begin{aligned} \frac{\partial}{\partial t}(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \varepsilon^3 \mathbf{u}_3) = (\widehat{L}_0 + \mu \widehat{L}_1)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \varepsilon^3 \mathbf{u}_3) + \\ (M_0)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2)(\varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2) + \\ (N_0)(\varepsilon \mathbf{u}_1)(\varepsilon \mathbf{u}_1)(\varepsilon \mathbf{u}_1) \end{aligned}$$

On the left-hand side, there will be products with increasing values of ε^n (where n is a positive integer). On the right-hand side, this will also be true, so, for all terms of the same ε^n on the right-hand side, they must be equal to that term on the left-hand side. Performing this expansion, and then setting all terms with the same ε equal to each other produces the following:

$$\frac{\partial \mathbf{u}_1}{\partial t} = \widehat{L}_0 \mathbf{u}_1$$

$$\frac{\partial \mathbf{u}_2}{\partial t} = \widehat{L}_0 \mathbf{u}_2 + M_0 \mathbf{u}_1 \mathbf{u}_1$$

$$\frac{\partial \mathbf{u}_3}{\partial t} = \widehat{L}_0 \mathbf{u}_3 + \chi \widehat{L}_1 \mathbf{u}_1 + 2M_0 \mathbf{u}_1 \mathbf{u}_2 + N_0 \mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_1$$

In perturbation theory, it is customary to produce a solution hierarchy. That is to say, the previous equations should be manipulated so that they have a common form, and, that the solution of each increasing term depends on the previous term. So, the solution hierarchy is:

$$(\frac{\partial}{\partial t} - \widehat{L}_0)\mathbf{u}_1 = 0 \tag{12}$$

$$(\frac{\partial}{\partial t} - \widehat{L}_0)\mathbf{u}_2 = M_0 \mathbf{u}_1 \mathbf{u}_1 \tag{13}$$

$$(\frac{\partial}{\partial t} - \widehat{L}_0)\mathbf{u}_3 = \chi L_1 \mathbf{u}_1 + 2M_0 \mathbf{u}_1 \mathbf{u}_2 + N_0 \mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_1 \tag{14}$$

The solution to equation (12) is required for equation (13), and this is true of (13) for (14). In addition, the difficulty of finding each solution increases. Apart from diminishing accuracy, this is another reason to limit the number of terms in the perturbation expansion. In order to find the solution to $\mathbf{u}_1$, this equation can be treated as an ordinary differential equation (ODE) and then solved with direct integration, as shown below:

$$(\frac{\partial}{\partial t} - \widehat{L}_0)\mathbf{u}_1 = 0$$

$$\frac{\partial \mathbf{u}_1}{\partial t} = \widehat{L}_0 \mathbf{u}_1$$

$$\int \frac{\partial \mathbf{u}_1}{\mathbf{u}_1} = \int \widehat{L}_0 \partial t$$

$$\mathbf{u}_1 = W e^{\widehat{L}_0 t}$$

Then, using the property that $e^{At} = U e^{\lambda t}$

$$\mathbf{u}_1 = W \mathbf{U} e^{\lambda t}$$

It is likely that λ is complex, so it will be assumed that λ has the form $\lambda = i\omega_0$, which results in a final answer of

$$\mathbf{u}_1 = W\mathbf{U}e^{i\omega_0 t} + W\overline{\mathbf{U}}e^{-i\omega_0 t} \tag{15}$$

where $\overline{U}$ is the complex conjugate of U.

Now that u_1 has a solution, u_2 can be derived. However, u_2 cannot be solved via direct integration, so a different method must be used. First, multiplying out $M_0\mathbf{u}_1\mathbf{u}_1$ results in the expression

$$M_0(\mathbf{U}\mathbf{U}W^2e^{2i\omega_0 t} + \overline{\mathbf{U}\mathbf{U}}W^2e^{-2i\omega_0 t} + 2W^2\mathbf{U}\overline{\mathbf{U}}) \tag{16}$$

Then, the left-hand side can be easily seen to be

$$\frac{\partial \mathbf{u}_2}{\partial t} - \widehat{L}_0\mathbf{u}_2 \tag{17}$$

In order to solve this, it is easiest to guess the form of the solution, so, the form of $\mathbf{u}_2$ will be assumed to be

$$\mathbf{u}_2 = A_1e^{ibt} + A_2e^{-ibt} + c \tag{18}$$

Now, using the method of judicious guessing, the partial derivative must also be derived. This is so that both the assumed solution and its derivative can be substituted into equation (17).

$$\frac{\partial \mathbf{u}_2}{\partial t} = A_1ibe^{ibt} - A_2ibe^{-ibt} \tag{19}$$

Now, equations (18) and (19) can be plugged into (17). Since equation (16) is the right-hand side, these expressions can be set equal to each other, producing:

$$A_1ibe^{ibt} - A_2ibe^{-ibt} - \widehat{L}_0(A_1e^{ibt} + A_2e^{-ibt} + c) = M_0(\mathbf{U}\mathbf{U}W^2e^{2i\omega_0 t} + \overline{\mathbf{U}\mathbf{U}}W^2e^{-2i\omega_0 t} + 2W^2\mathbf{U}\overline{\mathbf{U}}) \tag{20}$$

Each part of the solution can be treated individually

$$A_1 e^{ibt}[-(\widehat{L}_0 - ib)] = M_0 \mathbf{U}\mathbf{U}W^2 e^{2i\omega_0 t} \tag{21}$$

$$A_2 e^{-ibt}[-(\widehat{L}_0 + ib)] = M_0 \overline{\mathbf{U}\mathbf{U}}W^2 e^{2i\omega_0 t} \tag{22}$$

$$\widehat{L}_0 c = -2M_0 \mathbf{U}\overline{\mathbf{U}}W^2 \tag{23}$$

Equation (23) can be solved directly to produce

$$c = -2\widehat{L}_0^{-1} M_0 \mathbf{U}\overline{\mathbf{U}}W^2 \tag{24}$$

Equations (21) and (22) are solved in the same, so only (21) will be demonstrated:

$$A_1 e^{ibt} = -(\widehat{L}_0 - ib)^{-1} M_0 \mathbf{U}\mathbf{U}W^2 e^{2i\omega_0 t} \tag{25}$$

From this expression, it is easy to see that $A_1 = -(\widehat{L}_0 - ib)^{-1} M_0 \mathbf{U}\mathbf{U}W^2$ and $b = 2\omega_0$. For readability, V_+ and V_0 are defined such that $V_+ = -(\widehat{L}_0 - 2i\omega_0)^{-1} M_0 \mathbf{U}\mathbf{U}$ and $V_0 = -2\widehat{L}_0^{-1} M_0 U\overline{U}$ [10]. Thus, overall the solution of $\mathbf{u}_2$ is

$$\mathbf{u}_2 = V_+ W^2 e^{2i\omega_0 t} + c.c. + V_0 |W|^2 \tag{26}$$

where c.c. is the complex conjugate.

The solution of $\mathbf{u}_3$ begins much the same way as u_2. Recall the solution hierarchy for $\mathbf{u}_3$:

$$(\frac{\partial}{\partial t} - \widehat{L}_0)\mathbf{u}_3 = \chi L_1 \mathbf{u}_1 + 2M_0 \mathbf{u}_1 \mathbf{u}_2 + N_0 \mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_1$$

It is useful to do each term separately, as these calculations can be quite messy, so the equation will be further broken down into the following:

$$\chi L_1 \mathbf{u}_1 \tag{27}$$

$$2M_0 \mathbf{u}_1 \mathbf{u}_2 \tag{28}$$

$$N_0 \mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_1 \tag{29}$$

Equation (27) is trivial, so the following shows equation (28)

$$\mathbf{u}_1\mathbf{u}_2 = (W\mathbf{U}e^{i\omega_0 t} + W\overline{\mathbf{U}}e^{-i\omega_0 t})(V_+W^2e^{2i\omega_0 t} + V_-W^2e^{-2i\omega_0 t} + V_0|W|^2)$$

This expression can simply be multiplied through, however, all terms of $e^{ni\omega_0 t}$ where $|n| > 1$ will be labeled as higher harmonic terms (H.H.) and be kept separate. The RG method focuses on reducing equations that are too difficult (or impossible) to solve analytically, so there is no reason to include higher order terms. Higher order terms often make the equation just as, or in some cases even more difficult, to find solutions to. These terms add little to the accuracy of the final model, so at some point they will vanish from the equation.

Thus, with this in mind, equation (28) simplifies to

$$2M_0V_+W\overline{\mathbf{U}}e^{i\omega_0 t} + 2M_0V_-W\mathbf{U}e^{-i\omega_0 t} + H.H. \tag{30}$$

Expression (29) is handled in the same way, and results in

$$3N_0|W|^2W\mathbf{U}\mathbf{U}\overline{\mathbf{U}}e^{i\omega_0 t} + 3N_0|W|^2W\mathbf{U}\overline{\mathbf{U}\mathbf{U}}e^{-i\omega_0 t} + H.H. \tag{31}$$

Thus, the original equation for u_3 can be simplified to the following:

$$(\frac{\partial}{\partial t} - \widehat{L}_0)\mathbf{u}_3 = [\chi L_1WU + (2M_0\overline{\mathbf{U}}V_+ + 3N_0\mathbf{U}\mathbf{U}\overline{\mathbf{U}})|W|^2W]e^{i\omega_0 t} + c.c. + H.H. \tag{32}$$

To simplify even more, let $A = \chi L_1WU + (2M_0\overline{\mathbf{U}}V_+ + 3N_0\mathbf{U}\mathbf{U}\overline{\mathbf{U}})|W|^2W$ so that

$$(\frac{\partial}{\partial t} - \widehat{L}_0)\mathbf{u}_3 = Ae^{i\omega_0 t} + c.c. + H.H. \tag{33}$$

In order to get the most general form, let

$$A = \sum_\alpha A_\alpha \mathbf{U}_\alpha$$

This means that A can be expanded to create a more general solution. Just as in the case of ODEs, these solutions will be multiplied by the variable so that they're unique, however, since u_3 is a PDE, each solution will be multiplied by a factor of both t and x. To prevent secular terms, constants δ and δ' are introduced so that when $t = t_0$ and $x = x_0$ the secular terms of the first solution vanish.

$$\mathbf{u}_3 = \Big[A_1 c_1 (t - t_0 + \delta) - \frac{c_2}{2}\mathbf{D^{-1}}(x^2 - x_0^2 + \delta')\mathbf{U} + \sum_{\alpha \neq 1} \frac{A_\alpha}{i\omega_0 - \lambda_0^\alpha}\mathbf{U}_\alpha\Big] e^{i\omega_0 t} + c.c. + H.H. \quad (34)$$

The values of the introduced constants c_1 and c_2 are constrained by the condition that $c_1 + c_2 = 1$. With this result, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ have solutions. Since u is expressed as a perturbed value, adding together all the solutions creates an accurate approximation of the original equations. However, because this is an approximation, only terms of $e^{i\omega_0 t}$ are included, as larger terms are hard to work with and do not greatly affect the accuracy of the model. Thus, the solution of $\mathbf{u}_2$ is a higher harmonic and will vanish in the final result.

So, this means the approximation of $\mathbf{u}$ is

$$\mathbf{u} = \{\varepsilon W\mathbf{U} + \varepsilon^3[A_1 c_1 (t - t_0 + \delta) - \frac{c_2}{2}\mathbf{D^{-1}}(x^2 - x_0^2 + \delta')\mathbf{U} + \sum_{\alpha \neq 1} \frac{A_\alpha}{i\omega_0 - \lambda_0^\alpha}\mathbf{U}_\alpha]\} e^{i\omega_0 t} + c.c. + H.H. \quad (35)$$

The renormalization group method takes advantages of (or rather, is defined by) the property that:

$$\left.\frac{\partial \mathbf{u}}{\partial t_0}\right|_{t_0 = t} = 0 \quad (36)$$

$$\left.\frac{\partial \mathbf{u}}{\partial x_0}\right|_{x_0 = x} = 0 \quad (37)$$

These results follow directly from the fundamental theorem of envelops.

In order to use equations (36) and (37) it is important to mention that the complex conjugate and the higher harmonics are additional solutions to the system, meaning that they can be excluded from the application of the partial derivatives without any loss to the generality. Additionally, A_α will be treated as independent from x and t. Thus, after applying the partial derivatives we achieve the following:

$$\frac{\partial W}{\partial t} = \varepsilon^2 c_1 A_1 + O(\varepsilon^3) \tag{38}$$

$$\mathbf{D}\frac{\partial W}{\partial x} = -\varepsilon^2 x c_2 A_1 + O(\varepsilon^3) \tag{39}$$

Taking the partial derivative of (39) again yields a nearly symmetrical equation to (38)

$$\mathbf{D}\frac{\partial^2 W}{\partial x^2} = -\varepsilon^2 c_2 A_1 + O(\varepsilon^3) \tag{40}$$

The symmetry between equations (39) and (41) suggests taking advantage of the fact that $c_1 + c_2 = 1$. By subtracting the equations, the following takes place:

$$\frac{\partial W}{\partial t} - \mathbf{D}\frac{\partial^2 W}{\partial x^2} = \varepsilon^2 c_1 A_1 + O(\varepsilon^3) - (-\varepsilon^2 c_2 A_1 + O(\varepsilon^3))$$

$$\frac{\partial W}{\partial t} - \mathbf{D}\frac{\partial^2 W}{\partial x^2} = \varepsilon^2 c_1 A_1 + O(\varepsilon^3) + \varepsilon^2 c_2 A_1 - O(\varepsilon^3)$$

$$\frac{\partial W}{\partial t} - \mathbf{D}\frac{\partial^2 W}{\partial x^2} = \varepsilon^2 A_1 (c_1 + c_2) + [O(\varepsilon^3) - O(\varepsilon^3)]$$

which finally simplifies to:

$$\frac{\partial W}{\partial t} - \mathbf{D}\frac{\partial^2 W}{\partial x^2} = \varepsilon^2 A_1 \tag{41}$$

where

$$A_1 = \chi \mathbf{U}^* L_1 W U + (2\mathbf{U}^* M_0 \overline{\mathbf{U}} V_+ + 3\mathbf{U}^* N_0 \mathbf{U}\mathbf{U}\overline{\mathbf{U}})|W|^2 W \tag{42}$$

and $\mathbf{U}^*$ is the complex conjugate of $\mathbf{U}$.

Now that a generalized equation has been derived, all that's needed is to find all constants and perform some matrix operations. After they are derived, it is simply a matter of substituting and simplifying. L_0, while only in the V_+ term, is still important because $\mathbf{U}$ is defined as the eigenvector of L_0. This means that $\mathbf{U}$, $\overline{\mathbf{U}}$, and $\mathbf{U}^*$ all depend on L_0. Thus, the complete list of needed matrices is L_0, L_1, $\mathbf{U}$, $\overline{\mathbf{U}}$, $\mathbf{U}^*$, and N_0. M_0 also needs to be fine, but is dependent upon the vectors to the right of it, so it will be treated differently.

M_0 is a confusing constant to calculate, as it is not calculated by itself. Instead, it is calculated using the two matrices next to it as its components. Thus, it is calculated using:

$$(\mathbf{M_0AB}) = \begin{pmatrix} \frac{1}{2}\frac{\partial^2 H_1}{\partial\zeta^2}\Big|_{\zeta_0} A_1^2 + \frac{1}{2}\left(\frac{\partial^2 H_1}{\partial\zeta\partial\eta}\Big|_{\zeta_0} A_1B_2 + \frac{\partial^2 H_1}{\partial\eta\partial\zeta}\Big|_{\zeta_0} A_2B_1\right) + \frac{1}{2}\frac{\partial^2 H_1}{\partial\eta^2}\Big|_{\zeta_0} B_2^2 \\ \frac{1}{2}\frac{\partial^2 H_2}{\partial\zeta^2}\Big|_{\zeta_0} A_1^2 + \frac{1}{2}\left(\frac{\partial^2 H_2}{\partial\zeta\partial\eta}\Big|_{\zeta_0} A_1B_2 + \frac{\partial^2 H_2}{\partial\eta\partial\zeta}\Big|_{\zeta_0} A_2B_1\right) + \frac{1}{2}\frac{\partial^2 H_2}{\partial\eta^2}\Big|_{\zeta_0} B_2^2 \end{pmatrix} \tag{43}$$

It is important to note that $\mathbf{A}$ and $\mathbf{B}$ are 2x1 matrices. A_1 (not to be confused with the A_1 term in u) is the first component of $\mathbf{A}$ and A_2 is the second component. The same is true for $\mathbf{B}$. So, to calculate the whole matrix, $(M_0\mathbf{AB})$, the matrices immediately to the right of M_0 are crucial for its construction. N_0 is constructed in a similar way, however, it involves taking third order partials. Since the equation being used only has terms that go up to the second order, it is easily seen that the whole N_0 term is zero, and as such is disregarded in further calculations. Taking this into consideration produces a simplified A_1 equation where:

$$A_1 = \chi\mathbf{U}^*L_1UW + 2\mathbf{U}^*M_0\overline{\mathbf{U}}V_+|W|^2W \tag{44}$$

In order to calculate L_0 and L_1, the coefficients of the linear terms of equations (3) and (4) are extracted so that they create a 2x2 matrix L such that

$$L = \begin{pmatrix} \frac{-\alpha\beta}{\alpha\beta+1} & \frac{-\alpha}{\alpha\beta+1} \\ \frac{2\alpha\beta+1}{\alpha\beta+1} & \frac{-\alpha^2\beta}{\alpha\beta+1} \end{pmatrix} \tag{45}$$

The expression $\frac{1}{\alpha\beta+1}$ can be expanded using a geometric series:

$$L = \begin{pmatrix} -\alpha\beta(1-\alpha\beta+...) & -\alpha(1-\alpha\beta+...) \\ (2\alpha\beta+1)(1-\alpha\beta+...) & -\alpha^2\beta(1-\alpha\beta+...) \end{pmatrix} \tag{46}$$

Recalling that $L = L_0 + \mu L_1 + ...$, L_0 can be thought of as L when $\beta = 0$. This results in

$$L_0 = \begin{pmatrix} 0 & -\alpha \\ 1 & 0 \end{pmatrix} \tag{47}$$

L_1 can be found with a similar approach. Instead of all β terms being set to zero, in this case, all β terms except β^1 will be defined as zero, including β^0. This is equivalent to taking the coefficients of all β^1 terms of L. This produces the matrix

$$L_1 = \begin{pmatrix} -\alpha & -\alpha^2 \\ \alpha & -\alpha^2 \end{pmatrix} \tag{48}$$

Now, $\mathbf{U}$ can be calculated fairly quickly. First, find the eigenvalues of L_0 using the standard method, which turns out to simply be $\lambda = i\sqrt{\alpha}$. This also means that $\omega_0 = \sqrt{\alpha}$. $\mathbf{U}$ can then be found by solving the following system of equations for a and b:

$$L_0 \begin{pmatrix} a \\ b \end{pmatrix} = i\sqrt{\alpha} \begin{pmatrix} a \\ b \end{pmatrix} \tag{49}$$

It is easily seen, using basic properties of matrices and algebra, that $a = i\sqrt{\alpha}$ and $b = 1$. One thing to note is that the eigenvector is not unique. It can be scaled up or down and still be a solution; for reasons that will soon be justified, the eigenvector will be scaled by half, resulting in a final matrix:

$$\mathbf{U} = \frac{1}{2} \begin{pmatrix} i\sqrt{\alpha} \\ 1 \end{pmatrix} \tag{50}$$

Now this is used to find $\mathbf{U}^*$ and $\overline{\mathbf{U}}$. Beginning with $\mathbf{U}^*$, it is found by using the property that $\mathbf{U}^*\mathbf{U} = 1$. Choosing to scale $\mathbf{U}$ by $\frac{1}{2}$ allows us to choose a much more simple $\mathbf{U}^*$ since it

is not unique. If $\mathbf{U}$ had not been rescaled, $\mathbf{U}^*$ would be more complicated and not work well with the system. Thus, $\mathbf{U}^*$ is

$$\mathbf{U}^* = \frac{1}{i\sqrt{\alpha}} \begin{pmatrix} 1 & i\sqrt{\alpha} \end{pmatrix} \tag{51}$$

Then, since $\overline{\mathbf{U}}$ is the complex conjugate of $\mathbf{U}$, $\overline{\mathbf{U}}$ is shown to simply be

$$\overline{\mathbf{U}} = \frac{1}{2} \begin{pmatrix} -i\sqrt{\alpha} & 1 \end{pmatrix} \tag{52}$$

With the exception of the M_0 terms, all of the constants have been found, and that results in the following equation:

$$A_1 = \chi \frac{1}{2i\sqrt{\alpha}} \begin{pmatrix} 1 & i\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} -\alpha & \alpha^2 \\ \alpha & -\alpha^2 \end{pmatrix} \begin{pmatrix} i\sqrt{\alpha} \\ 1 \end{pmatrix} W + \frac{2}{i\sqrt{\alpha}} \begin{pmatrix} 1 & i\sqrt{\alpha} \end{pmatrix} \left(M_0 \overline{\mathbf{U}} V_+ \right) |W|^2 W \tag{53}$$

Equation (53) has two terms; the first of which is very easy to calculate, the second of which is much more tedious. The first one can be solved the following way:

$$\chi \frac{1}{2i\sqrt{\alpha}} \begin{pmatrix} 1 & i\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} -i\alpha\sqrt{\alpha} + \alpha^2 \\ +i\alpha\sqrt{\alpha} - \alpha^2 \end{pmatrix}$$

$$\chi \frac{1}{2i\sqrt{\alpha}} \left(-i\alpha^2\sqrt{\alpha} - i\alpha\sqrt{\alpha} \right)$$

which can be factored and simplified to the following:

$$-\chi \frac{\alpha}{2}(\alpha + 1) \tag{54}$$

In order to simplify the second term, V_+ must be calculated separately. Recall:

$$V_+ = -(\widehat{L}_0 - 2i\omega_0)^{-1} M_0 \mathbf{U}\mathbf{U}$$

The inverse matrix must be dealt with, and the first thing that needs to be done is to add $-2i\omega_0$ to the L_0 matrix. This can be done by multiplying $-2i\omega_0$ by the identity matrix like so:

$$\begin{pmatrix} 0 & -\alpha \\ 1 & 0 \end{pmatrix} - 2i\sqrt{\alpha}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The scalar can then be multiplied through, and the matrices can be added component wise, which results in the matrix:

$$\begin{pmatrix} -2i\sqrt{\alpha} & -\alpha \\ 1 & -2i\sqrt{\alpha} \end{pmatrix}$$

This matrix will be defined as S_0. The inversion of S_0 is the matrix that makes the following expression true:

$$S_0 S_0^{-1} = \mathbf{1}$$

One way to approach this is to insert variables into S_0^{-1} and solve it as a system of equations, which looks like the following:

$$\begin{pmatrix} -2i\sqrt{\alpha} & -\alpha \\ 1 & -2i\sqrt{\alpha} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This produces the following system of equations:

$$-2ia\sqrt{\alpha} - \alpha c = 1 \tag{55}$$

$$a - wic\sqrt{\alpha} = 0 \tag{56}$$

$$-2ib\sqrt{\alpha} - \alpha d = 0 \tag{57}$$

$$b - wid\sqrt{\alpha} = 1 \tag{58}$$

These are very easily solved to find that:

$$a = \frac{2i\sqrt{\alpha}}{3\alpha} \tag{59}$$

$$b = -\frac{1}{3} \tag{60}$$

$$c = \frac{1}{3\alpha} \tag{61}$$

$$d = \frac{2i\sqrt{\alpha}}{3\alpha} \tag{62}$$

Equations (60), (61), (62), and (63) can be inserted into S_0^{-1} to get that:

$$S_0^{-1} = \begin{pmatrix} \frac{2i\sqrt{\alpha}}{3\alpha} & -\frac{1}{3} \\ \frac{1}{3\alpha} & \frac{2i\sqrt{\alpha}}{3\alpha} \end{pmatrix} \tag{63}$$

Finally, the negative can be distributed to all the components of S_0 to produce the matrix:

$$\begin{pmatrix} -\frac{2i\sqrt{\alpha}}{3\alpha} & \frac{1}{3} \\ -\frac{1}{3\alpha} & -\frac{2i\sqrt{\alpha}}{3\alpha} \end{pmatrix} \tag{64}$$

Now, $(M_0\mathbf{U}\mathbf{U})$ still needs to be calculated. Using equation (43) with $A = \mathbf{U}$ and $B = \mathbf{U}$ will result in the following:

$$\begin{pmatrix} \frac{-i\sqrt{\alpha}}{4} \\ \frac{i\sqrt{\alpha}}{4} \end{pmatrix} \tag{65}$$

Therefore, combining (64) and (65):

$$v_+ = \begin{pmatrix} -\frac{2i\sqrt{\alpha}}{3\alpha} & \frac{1}{3} \\ -\frac{1}{3\alpha} & -\frac{2i\sqrt{\alpha}}{3\alpha} \end{pmatrix} \begin{pmatrix} \frac{-i\sqrt{\alpha}}{4} \\ \frac{i\sqrt{\alpha}}{4} \end{pmatrix} \tag{66}$$

which ultimately simplifies to

$$V_+ = \begin{pmatrix} \frac{i\alpha\sqrt{\alpha}-2\alpha}{12\alpha} \\ \frac{i\sqrt{\alpha}+2\alpha}{12\alpha} \end{pmatrix} \tag{67}$$

With this, it is now possible to calculate $M_0\overline{\mathbf{U}}V_+$. Using the same formula, except with $A = \overline{\mathbf{U}}$ and $B = V_+$, the following is produced:

$$(M_0\overline{\mathbf{U}}V_+) = \begin{pmatrix} \frac{1}{48} + \frac{i\sqrt{\alpha}}{48} \\ -\frac{1}{48} - \frac{i\sqrt{\alpha}}{48} \end{pmatrix} \tag{68}$$

Plugging this into the overall expression, produces this:

$$\frac{1}{24i\sqrt{\alpha}} \begin{pmatrix} 1 & i\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} 1 + i\sqrt{\alpha} \\ 1 - i\sqrt{\alpha} \end{pmatrix} \tag{69}$$

This expression can be simplified to

$$-i\frac{(\alpha+1)}{24\sqrt{\alpha}} \tag{70}$$

Combining (54) and (71) results in this final equation:

$$\frac{\partial W}{\partial t} - \mathbf{D}\frac{\partial^2 W}{\partial x^2} = \varepsilon^2\left[-\chi\frac{\alpha}{2}(\alpha+1)W - i\frac{(\alpha+1)}{24\sqrt{\alpha}}]|W|^2W\right] \tag{71}$$

Equation (72) is in the form of the complex Ginzburg-Landau equation, which is an equation that has well-understood wave solutions. In order to find these solutions, equation (72) must be slightly reorganized. The desired form is:

$$\frac{\partial W}{\partial t} = (1 - ic_1|W|^2)W + (1 + ic_2)\frac{\partial^2 W}{\partial x^2} \tag{72}$$

This new form is not a drastic change, but it does introduce new constants, namely c_1 and c_2 which are not to be confused with the c_1 and c_2 from equation (34). After adding $\mathbf{D}\frac{\partial^2 W}{\partial x^2}$ to the other side, the other terms can be factored such that the new constants must satisfy the following conditions:

$$\mathbf{D} = 1 + ic_1 \tag{73}$$

$$1 - ic_2|W|^2 = -\varepsilon^2\chi\frac{\alpha}{2}(\alpha+1) - i\varepsilon^2\frac{(\alpha+1)}{24\sqrt{\alpha}}|W|^2 \tag{74}$$

Equation (74) is in a form that is solvable with standard algebra, whereas equation (75) has potential problems. If traditional algebra were used to solve for C_2, then C_2 would contain $|W|^{-2}$ which is not desirable since W is the solution. To avoid this, it must then be assumed that

$$1 = -\varepsilon^2\chi\frac{\alpha}{2}(\alpha+1) \tag{75}$$

$$c_2 = \varepsilon^2\frac{(\alpha+1)}{24\sqrt{\alpha}} \tag{76}$$

Using this assumption, c_1 and C_2 are shown to be

$$c_1 = -i(D-1) \tag{77}$$

$$c_2 = \varepsilon^2 \frac{(\alpha+1)}{24\sqrt{\alpha}} \tag{78}$$

The CGLE has known wave solutions of the form:

$$W_Q = R_Q e^{i(Qt-\omega_Q t)} \tag{79}$$

where

$$R_Q = \sqrt{1-Q^2} \tag{80}$$

$$\omega_Q = c_1 Q^2 - (1-Q^2)c_2 \tag{81}$$

$$|Q| < 1 \tag{82}$$

With these parameters, the wave solution takes the form:

$$W_Q(x,t) = R_Q e^{i\{Qx-[(-i(D-1))Q^2-(1-Q^2)\frac{\varepsilon^2(\alpha+1)}{24\sqrt{\alpha}}]t\}} \tag{83}$$

Thus, the behavior of the solutions to the equations presented at the beginning of this section can be studied by studying equation (84). Since the CGLE is known to have bifurcation, it is useful to know when the solutions transition from stability to instability. There are several ways of finding these conditions, but one known condition is that if

$$Q^2 < \frac{1+c_1c_2}{2c_2^2+c_1c_2+3} \tag{84}$$

holds, then the solution is long-wave stable. Thus, whenever this condition is not met, the solution is unstable.

Plugging equations (78) and (79) into equation (85) will allow for investigation into the stability of the specific equations used so far. These substitutions produce

$$Q^2 < \frac{1+\varepsilon^2(-i[\mathbf{D}+i])(\frac{\alpha+1}{24\sqrt{\alpha}})}{\varepsilon^4(\frac{(\alpha+1)^2}{576\alpha})+\varepsilon^2(-i[\mathbf{D}+i])(\frac{\alpha+1}{24\sqrt{\alpha}})+3} \tag{85}$$

There are a few problems with this expression, namely that there is a matrix $\mathbf{D}$, and also complex numbers. This is a problem because complex numbers are not ordered, thus it is not possible to compare sizes. There is one obvious way to deal with this problem, and one way that requires some arbitrary decisions. Beginning with the intuitive nature, ε represents a small number between 0 and 1. Thus, for a sufficiently small ε, the equation can be reduced to

$$Q^2 < \frac{1}{3} \tag{86}$$

which is known as the Eckhaus condition. Alternative, if ε is not small enough that such a condition is met, then other assumptions will have to be made.

For the sake of demonstrating how the stability of the system might change, naive and arbitrary assumptions about the system are made. Since ε is not small, $\varepsilon = 1$ is not an absurd assumption. This is completely arbitrary, as any small real value could have been chosen instead. Next, in order to deal with both the matrix and the complex numbers it will be assumed that $D = 1$. Since $\mathbf{D}$ is a diagonalized matrix whose values are constant, $\mathbf{D}$ rescales but does not affect the nature of the solution, thus this assumption may change the scale but will not affect the overall behavior. Plugging these values into equation (86) and performing simple algebra results in the simplified equation

$$Q^2 < \frac{288\alpha}{(\alpha+1)^2 + 864\alpha} \tag{87}$$

Using the fact that Q is some arbitrary real number, and that $|Q| < 1$, it is perfectly reasonable to pick Q such that $Q = 0.9$.

Then, the solution is stable so long as $-0.0019 < \alpha < -0.0012$. Since α is a constant, it can easily be picked such that the solution is stable. However, in nature, these values are not picked out of convenience. Instead, Q and α will vary as conditions around them change. This means that the stability of the solution will also change as the conditions change. While only one example was done here, it is easy to see how with the stability condition, the solution will often fluctuate between stable and unstable.

Conclusion

The renormalization group method is an incredibly useful tool in the process of reducing PDEs to more manageable forms. If given a system of PDEs that matches the general form of this paper, that system can be immediately reduced to the form of a CGLE. Additionally, the derivation of this fact is not nearly as difficult as other methods. The ability to use the method without as much specialization makes it much more viable for wide spread use and possibly the ability to handle a greater variety of equations.

As shown with the specific set of PDEs, it's easy to apply the reduction to even complicated sets of PDEs. Doing an example that is non-linear in nature is important to the demonstration of the method, as calculating some of the unique matrices is not intuitive. Moreover, previous literature has used confusing notation, and this paper attempts to improve both the clarity of the notation, along with provide clear example of application.

One final point: it is incredibly convenient that reaction-diffusion equations can be reduced to the CGLE. Since the CGLE has been well studied along with its known solutions, it is no longer difficult to find wave solutions to PDEs that originally were not solvable with known analytic methods. This means that many PDEs that were previously only studyable with standard numerical analysis can have more meaningful information inferred through the known wave equations of the CGLE.

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