The Zero Forcing Polynomial of a Ladder Graph

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Motivation

Zero forcing was first introduced in 2008, and has been studied in connection with quantum physics, theoretical computer science, power network monitoring, and linear algebra.

Definitions

- A graph G is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges (an edge is a two-element subset of vertices).
- The degree of a vertex v is the number of edges incident to the vertex.
- If G is a graph with each vertex colored either white or blue, the color change rule states that if v is a blue vertex of G with exactly one white neighbor u, then change the color of u to blue.

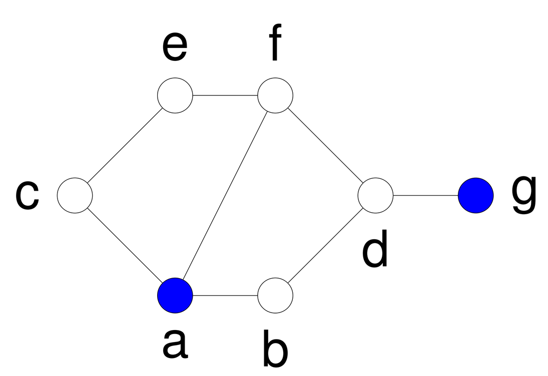


Figure: Graph with vertices a and g initially colored vertices blue.

- A zero forcing set for a graph G is a set of initially colored blue vertices $S \subseteq V(G)$ such that by applying the zero forcing color change rule repeatedly, all the remaining vertices eventually become blue.
- The zero forcing number of a graph G is the minimum cardinality over all zero forcing sets of G.

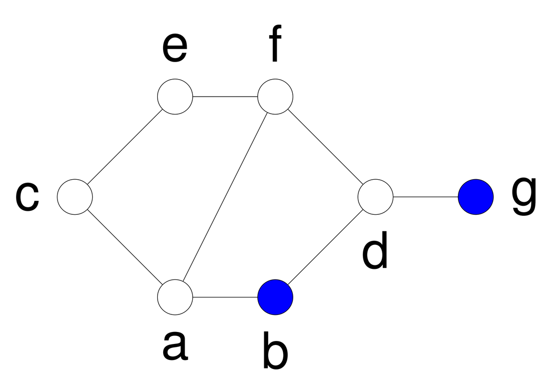


Figure: Zero forcing set of this graph is $\{b, g\}$, and the zero forcing number is 2.

- The zero forcing polynomial of a graph G is given by:

$$\mathcal{Z}(G; x) = \sum_{i=0}^{n} z(G; i)x^i$$

where $z(G; i)$ is the number of zero forcing sets of G of cardinality i.
- A two-dimensional n-ladder graph is the graph Cartesian product $P_2 \square P_n$, where P_n is the path graph on n vertices.

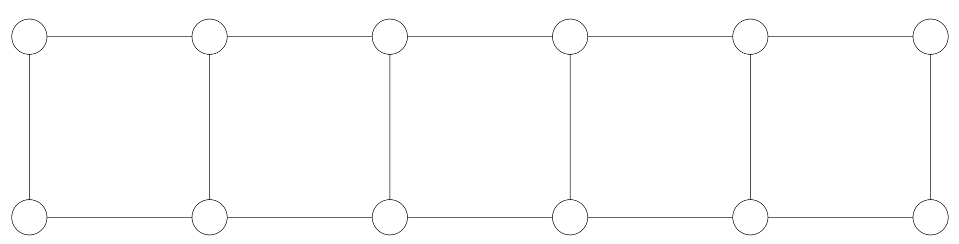


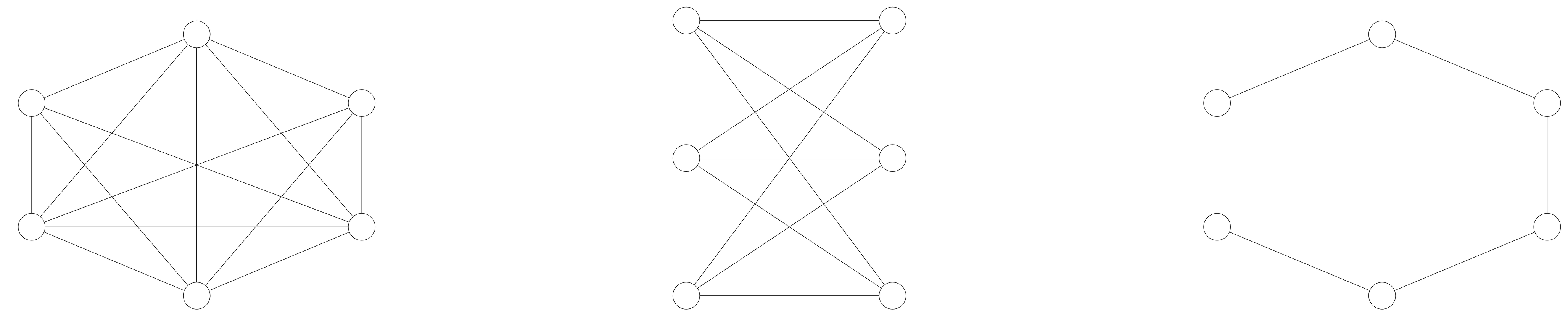
Figure: The graph of $P_2 \square P_6$.

- The corner vertex of an n-ladder graph is a vertex of degree 2.
- An internal vertex of a n-ladder graph is a vertex of degree 3.

Goal

Our research team aims to describe the closed formula for the zero forcing polynomial of n-ladder graphs.

Related Results



Complete graph K_6 | Complete bipartite graph $K_{3,3}$ | Cycle graph C_6

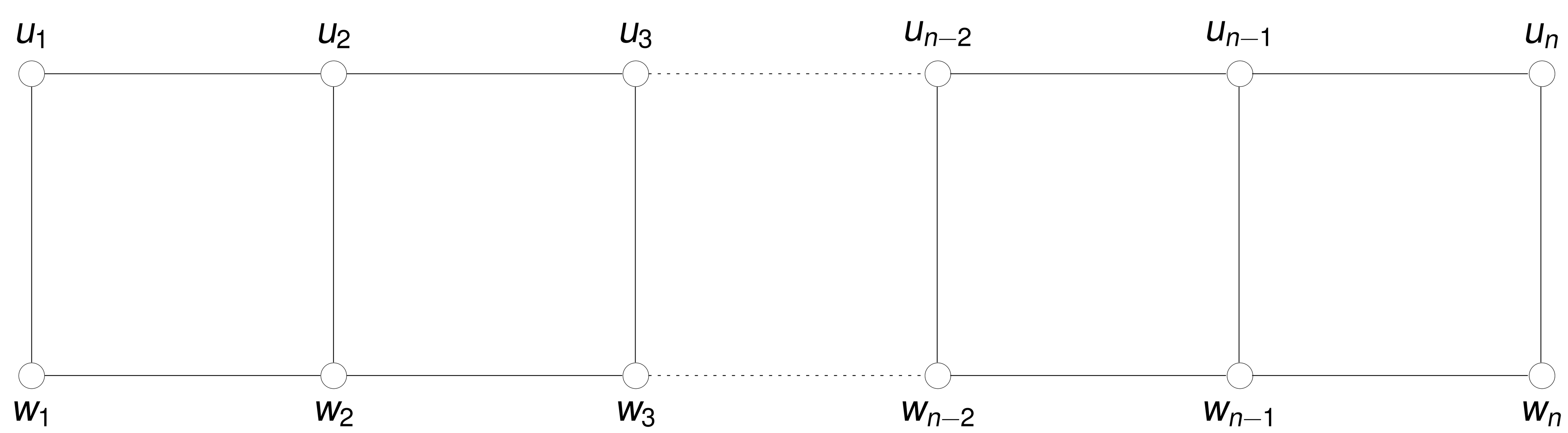
Theorem [2] Complete Graph: For $n \geqslant 2$, then $\mathcal{Z}(K_n; x) = x^n + nx^{n-1}$.

Theorem [2] Complete Multipartite Graph: If $a_1, \ldots, a_k \geqslant 2$, then

$$\mathcal{Z}(K_{a_1,\ldots,a_k}; x) = \left(\sum_{1 \leqslant i < j \leqslant k} a_i a_j\right)x^{n-2} + nx^{n-1} + x^n.$$

Theorem [2] Cycle Graph: For $n \geqslant 3$, then $\mathcal{Z}(C_n; x) = \sum_{i=2}^{n} \left(\binom{n}{i} - \frac{n}{i}\binom{n-i-1}{i-1}\right) x^i$.

Main Results



Theorem: Let G be the n-ladder graph $P_2 \square P_n$. Then $z(P_2 \square P_n; 2) = 6$ when $n \geqslant 3$.

- A zero forcing set of size 2 in $P_2 \square P_n$ must include a corner vertex.

Theorem: Let G be the n-ladder graph $P_2 \square P_n$. Then $z(P_2 \square P_n; 2n-3) = \binom{2n}{2n-3}$.

- Every subset of vertices of $P_2 \square P_n$ of size $2n-3$ is a zero forcing set.

Theorem: Let G be the n-ladder graph $P_2 \square P_n$. Then $z(P_2 \square P_n; 4) = \left(\binom{2n}{4} - 4\right) = 66$ when $n = 4$.

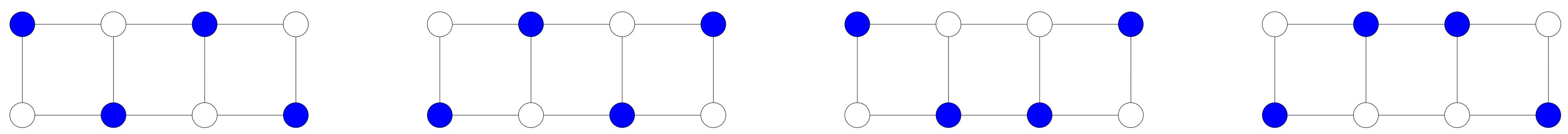
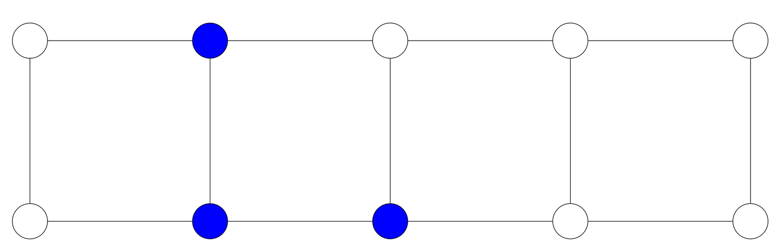


Figure: The set of blue vertices of size 4 that does not form a zero forcing set in $P_2 \square P_4$.

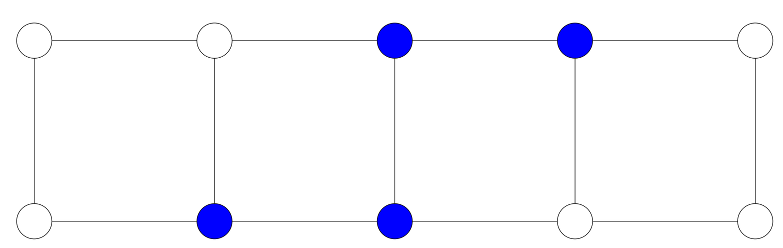
Other Results

Lemma: An L consists of three vertices u_i, w_i and either u_{i+1} or w_{i+1} for $i = 2$ or it consists of three vertices u_i, w_i and either u_{i-1} or w_{i-1} for $i = n-1$. A set that forms a L is a zero forcing set.

Lemma: A Z consists of four vertices $u_{i-1}, u_i, w_i, w_{i+1}$ or $u_{i+1}, u_i, w_i, w_{i-1}$ for $3 \leqslant i \leqslant n-2$. A set that forms a Z is a zero forcing set.



An L shape in $P_2 \square P_5$



A Z shape in $P_2 \square P_5$

Future Research

- **Conjecture:** For $n \geqslant 5$, $z(P_2 \square P_n; 4) = 4\binom{2n-2}{2} + 2\binom{2n-4}{2} + 19n - 82$.

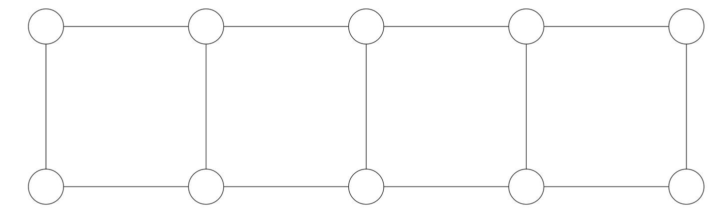


Figure: The graph of $P_2 \square P_5$.

- Establish the zero forcing polynomial for the n-ladder graph $P_2 \square P_n$.
- We hope to extend our research on ladder graphs to grid graphs. A two-dimensional $m \times n$ grid graph is the graph Cartesian product $P_m \square P_n$, where P_n is the path graph on n vertices.

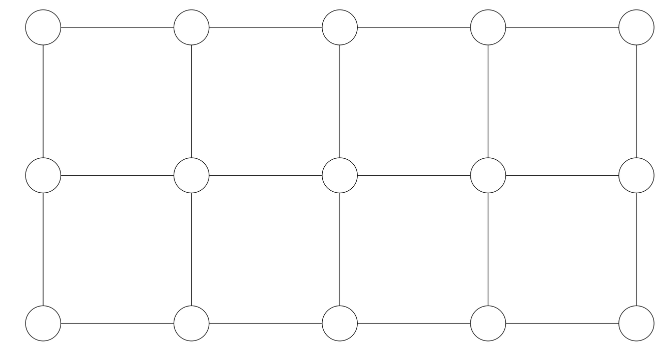


Figure: The graph of $P_3 \square P_5$.

Open Problems

- Characterize the zero forcing polynomials for other families of graphs.
- Determine which graphs are recognized by their zero forcing polynomials. It was shown by Boyer et al. [2] that path graphs and complete graphs are recognized by their zero forcing polynomials.
- **Conjecture** [2]: For any graph G on n vertices, $z(G; i) \leqslant z(P_n; i)$ for $1 \leqslant i \leqslant n$. That is, the zero forcing coefficient for a graph on n vertices is less than or equal to the zero forcing coefficient for the path graph on n vertices.

Acknowledgements

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2. K. Boyer, B. Brimkov, S. English, D. Ferrero, A. Keller, R. Kirsch, M. Phillips, and C. Reinhart. *The zero forcing polynomial of a graph*. arXiv: 1801.08910v1.