An Equivalence Between the Polytabloid Bases and Specht Polynomials for Irreducible Representations of the Symmetric Group

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1. Introduction

A Young tableau is a combinatorial object useful in the study of the representation theory of the symmetric group S_n and its properties. A **standard Young tableau** is a tableau where the entries are from {1,2,...,n} and where the entries across each row and down each column are increasing. A **tabloid** is an equivalence class of labelings of a Young tableaux of shape λ, where λ is an integer partition of n. The symmetric group S_n acts on a Young tableau (and its associated tabloid) by permuting the entries.

Let t be a **standard Young tableau** and $\{t\}$ its associated **tabloid**.

Example 1.

If $\lambda = (4, 2, 2, 1)$ then

$$t = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 \\ \cline{1-2} 6 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array} \quad \text{and } \{t\} = \begin{array}{cccc}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 \\ \hline 6 & 8 \\ \hline 9 \\ \hline\end{array}$$

Let V_t be the subgroup of S_n that preserves the columns of t, and set

$$\mathcal{V}_t = \sum_{\sigma \in V_t} \operatorname{sgn}(\sigma)\sigma$$

which is an element of the group algebra $\mathbb{C}(S_n)$. Let $\mathcal{T}$ be the $\mathbb{C}$-vector space of *polytabloids* with basis the set of tabloids with shape λ, and let $e_t = \mathcal{V}_t\{t\}$ be a vector in $\mathcal{T}$.

2. Introduction

By Theorem 2.5.2 from [1],

$$\{e_t : t \text{ is a standard } \lambda - \text{tableau}\}$$

is a basis for an irreducible representation S^λ of S_n.

Now let $\mathcal{P}(x_1, \dots, x_n) = \mathcal{P}$ be the vector space of polynomials in n variables and let S_n act on this space by permuting the variables. Let t be a standard Young Tableau with shape λ, let $t_{[1]}, t_{[2]}, ..., t_{[k]}$ be the columns of t, and we write $t = t_{[1]}t_{[2]}...t_{[k]}$. Let $t_{i,1}, t_{i,2}, ..., t_{i,l}$ be the entries of the the i^{th} column of t and let $\Delta(t_{[i]}) = \Delta\left(x_{t_{i,1}}, x_{t_{i,2}}, ..., x_{t_{i,l}}\right)$ be the **Vandermonde Determinant** in the variables sub-scripted by the entries from the i^{th} column of t.

Example 2.

$$\Delta(t_{[1]}) = \Delta(x_1, x_2, x_6, x_9) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_6 & x_9 \\ x_1{}^2 & x_2{}^2 & x_6{}^2 & x_9{}^2 \\ x_1{}^3 & x_2{}^3 & x_6{}^3 & x_9{}^3 \end{vmatrix}$$

Finally, let $\Delta(t)$ be the polynomial $\Delta(t) = \Delta(t_{[1]})\Delta(t_{[2]})...\Delta(t_{[k]})$.

In [2], it is shown that

$$\{\Delta(t) : \text{t is a standard } \lambda - \text{tableau}\}$$

is a basis for an **irreducible representation** of S_n in $\mathcal{P}$ that is equivalent to the representation S^λ in $\mathcal{T}$. The polynomials $\Delta(t)$ are called ***Specht*** polynomials.

3. Theorem

Let $\rho : \mathcal{T} \to \mathcal{P}$ be the map that takes the tableau t (or the tabloid $\{t\}$) to the monomial $m_t = \Pi_{\alpha=1}^n x_\alpha{}^{\theta(\alpha)}$ where $\theta(\alpha) = j - 1$ if α is in the j^{th} row of $\{t\}$.

Example 3.

$$\{t\} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \rho(\{t\}) = x_1{}^0 x_2{}^0 x_3{}^1 x_4{}^1 = m_t.$$

Note 1. *Since tabloids are row-equivalent, the map is independent of the representative t of $\{t\}$. Also, the map ρ intertwines the representations of S_n on $\mathcal{T}$ and $\mathcal{P}$,* i.e., *$\rho(\tau\{t\}) = \tau(\rho(\{t\}))$, for $\tau \in S_n$. It is sufficient to prove this for the case that τ is a two-cycle, since if τ interchanges the entries in two rows of $\{t\}$, it also interchanges the corresponding variables in the monomial m_t.*

Theorem 1.

$$\rho(e_t) = \Delta(t)$$

Remark 1. *We assume that this result is known, but it is not recorded in any of the standard references.*

Lemma 1. $\rho(t_{[1]}t_{[2]}...t_{[k]}) = \rho(t_{[1]})\rho(t_{[2]})...\rho(t_{[k]})$.

Proof. We prove this for the case for $t = t_{[1]}t_{[2]}$ and the result follows by induction.

$$\rho(t_{[1]}t_{[2]}) = \prod_{\alpha \in t_{[1]}t_{[2]}} x_\alpha^{\theta(\alpha)} = \prod_{r \in t_{[1]}, s \in t_{[2]}} x_r^{\theta(r)} x_s^{\theta(s)} = \prod_{r \in t_{[1]}} x_r^{\theta(r)} \prod_{s \in t_{[2]}} x_s^{\theta(s)} = \rho(t_{[1]})\rho(t_{[2]}).$$

□

4. Proof

Lemma 2. $V_t = V_{t_{[1]}} \times V_{t_{[2]}} \times ... \times V_{t_{[k]}}$ *as a direct product of subgroups of S_n.*

Proof. Clearly, the product $V_{t_{[1]}} \times V_{t_{[2]}} \times ... \times V_{t_{[k]}}$ is contained in V_t.

Now suppose $\sigma \in V_t$. Since σ can be written as a product of two-cycles, let $\sigma = \sigma_1\, \sigma_2 \dots \sigma_n$. Since σ only preserves columns, any two-cycles that preserve distinct columns must be disjoint. Since disjoint cycles commute, we can reorder the two-cycles in the above product σ so that each grouping preserves a distinct column. Finally, since $V_{t_{[r]}} \bigcap V_{t_{[s]}} = \{(1)\}$ for $r \neq s$, this product is direct.

□

Corollary 1. $\mathcal{V}_t = \mathcal{V}_{t_{[1]}} \times \mathcal{V}_{t_{[2]}} \times ... \times \mathcal{V}_{t_{[k]}}$, *the product on the right being taken in $\mathbb{C}(S_n)$.*

Since

$$\begin{aligned} \mathcal{V}_{t_{[1]}} \times \mathcal{V}_{t_{[2]}} &= \textstyle\sum_{\sigma_1 \in V_{t_{[1]}}} (\operatorname{sgn}(\sigma_1))\sigma_1 \cdot \sum_{\sigma_2 \in V_{t_{[2]}}} (\operatorname{sgn}(\sigma_2))\sigma_2 \\ &= \textstyle\sum_{\sigma_1 \in V_{t_{[1]}}, \sigma_2 \in V_{t_{[2]}}} (\operatorname{sgn}(\sigma_1))(\operatorname{sgn}(\sigma_2))\sigma_1\sigma_2 \\ &= \textstyle\sum_{\sigma_1 \in V_{t_{[1]}}, \sigma_2 \in V_{t_{[2]}}} (\operatorname{sgn}(\sigma_1\sigma_2))\sigma_1\sigma_2 \\ &= \textstyle\sum_{\sigma_1\sigma_2 \in V_{t_{[1]}} \times V_{t_{[2]}}} (\operatorname{sgn}(\sigma_1\sigma_2))(\sigma_1\sigma_2) \\ &= \mathcal{V}_{t_{[1]}t_{[2]}}. \end{aligned}$$

The result follows by induction.

Lemma 3. $\mathcal{V}_{t_{[r]}}(t_{[s]}) = t_{[s]}$ *if $s \neq r$, since by definition, $V_{t_{[r]}}$ permutes only the entries in the rth column. Equivalently, $\mathcal{V}_{t_{[r]}}(m_{t_{[s]}}) = m_{t_{[s]}}$ if $s \neq r$.*

5. Proof

Proof. We start with the case $t = t_{[1]}$. By the definition of the Vandermonde determinant and applying Leibnitz' formula for the determinant of an $n \times n$ matrix,

$$\Delta(t_{[1]}) = \sum_{\sigma \in V_{t_{[1]}}} \operatorname{sgn}(\sigma) \prod_{j=1}^{l} x_{\sigma(t_{1,j})}{}^{j-1}.$$

Thus

$$\begin{aligned} \rho\left(e_{t_{[1]}}\right) &= \rho\left(\mathcal{V}_{t_{[1]}} t_{[1]}\right) \\ &= \rho\left(\textstyle\sum_{\sigma \in V_{t_{[1]}}} \operatorname{sgn}(\sigma)\sigma\, t_{[1]}\right) \\ &= \textstyle\sum_{\sigma \in V_{t_{[1]}}} \operatorname{sgn}(\sigma) \prod x_{\sigma(t_{1,j})}{}^{j-1} \\ &= \Delta(t_{[1]}). \end{aligned}$$

Finally we combine these results:

$$\begin{aligned} \rho(e_t) &= \rho(\mathcal{V}_t\, t) && \text{by definition,} \\ &= \rho(\mathcal{V}_{t_{[1]}} \times \cdots \times \mathcal{V}_{t_{[k]}} t_{[1]} \cdots t_{[k]}) && \text{by corollary 1 and notation for } t, \\ &= \mathcal{V}_{t_{[1]}} \times \cdots \times \mathcal{V}_{t_{[k]}} \rho(t_{[1]} \cdots t_{[k]}) && \text{since } \rho \text{ intertwines,} \\ &= \mathcal{V}_{t_{[1]}} \rho(t_{[1]}) \cdots \mathcal{V}_{t_{[k]}} \rho(t_{[k]}) && \text{by lemmas 1 and 3,} \\ &= \rho(\mathcal{V}_{t_{[1]}} t_{[1]}) \cdots \rho(\mathcal{V}_{t_{[k]}} t_{[k]}) && \text{since } \rho \text{ intertwines,} \\ &= \rho(e_{t_{[1]}}) \cdots \rho(e_{t_{[k]}}) && \text{by definition,} \\ &= \Delta(t_{[1]}) \cdots \Delta(t_{[k]}) && \text{by the discussion above,} \\ &= \Delta(t) && \text{by definition.} \end{aligned}$$

□

6. Example

Let

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \text{ so that } \{t\} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}.$$

Then V_t={(1), (2 4), (1 3), (1 3)(2 4)} and so $\mathcal{V}_t$ = (1) - (1 3) - (2 4) + (1 3)(2 4).

Now if we apply $\mathcal{V}_t$ on {t}, we have

$$e_t = \mathcal{V}_t(\{t\}) = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} + \begin{array}{cc}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}.$$

Thus

$$\begin{aligned} \rho(\mathcal{V}_t(\{t\})) &= x_1{}^0 x_2{}^0 x_3{}^1 x_4{}^1 - x_2{}^0 x_3{}^0 x_1{}^1 x_4{}^1 - x_1{}^0 x_4{}^0 x_2{}^1 x_3{}^1 + x_3{}^0 x_4{}^0 x_1{}^1 x_2{}^1 \\ &= x_1x_2 + x_3x_4 - x_2x_3 - x_1x_4. \end{aligned}$$

And by definition of the Specht polynomial,

$$\begin{aligned} \Delta(t) &= \begin{vmatrix} 1 & 1 \\ x_1 & x_3 \end{vmatrix} \times \begin{vmatrix} 1 & 1 \\ x_2 & x_4 \end{vmatrix} \\ &= (x_1 - x_3) \times (x_2 - x_4) \\ &= x_1x_2 + x_3x_4 - x_2x_3 - x_1x_4. \end{aligned}$$

Remark 2. *This result generalizes. For example*

$$\widetilde{\rho}(\mathcal{V}_t(\{t\})) = x_1{}^1 x_2{}^1 x_3{}^3 x_4{}^3 - x_2{}^1 x_3{}^1 x_1{}^3 x_4{}^3 - x_1{}^1 x_4{}^1 x_2{}^3 x_3{}^3 + x_3{}^1 x_4{}^1 x_1{}^3 x_2{}^3.$$

References

[1] B. E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.* Springer, 2001.

[2] Specht. Die irreduziblen darstellungen der symmetrischen gruppe. *Math. Z.*, 39:696–711, 1935.

Acknowledgments

- Office of Research and Sponsored Programs, UW-Eau Claire
- Department of Mathematics, UW-Eau Claire
- Modified using LaTeX