Mirror Symmetry in Reflexive Polytopes

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Motivations

There are two main theories used by physicists to explain the inner-workings of the universe. *General relativity* is used to describe the very large, while *quantum mechanics* describes the very small. For decades physicists have sought after a so called *unified field theory* to combine these two models. Currently, the most widely accepted candidate for a unified field theory is known as *string theory*.

In order to reconcile general relativity with quantum mechanics, string theory extends our classical 4D model of space-time into extra dimensions. At every unique point in our known four dimensions, these extra dimensions have the structure of *Calabi-Yau varieties*, or 6D algebraic varieties. It turns out that there are always two Calabi-Yau varieties that produce a particular physical model. In mathematics we call this phenomenon *mirror symmetry*.

Reflexive Polytopes

The *polar duality* transformation takes a polytope with integer lattice points to its polar dual. Let Δ be a lattice polytope which contains the origin. The polar dual polytope Δ° is the polytope given by:

$$\{(m_1,\ldots,m_k) : (n_1,\ldots,n_k)\cdot(m_1,\ldots,m_k) \geq -1 \quad \forall\ (n_1,\ldots,n_k) \in \Delta\}$$

A lattice polytope is defined to be *reflexive* if its polar dual is also a lattice polytope. In addition, the polar dual of the polar dual of a reflexive polytope is the original polytope.

There are an infinite number of reflexive polytopes in any given dimension ≥ 3, but they can all be classified into different equivalence classes if we consider the linear transformations of reflecting, rotating, and shearing as mapping to the same class. Reflexive polytopes have been classified in 2D, 3D, and 4D, with 16, 4 319, and 473 800 776 classes of equivalent polytopes respectively.

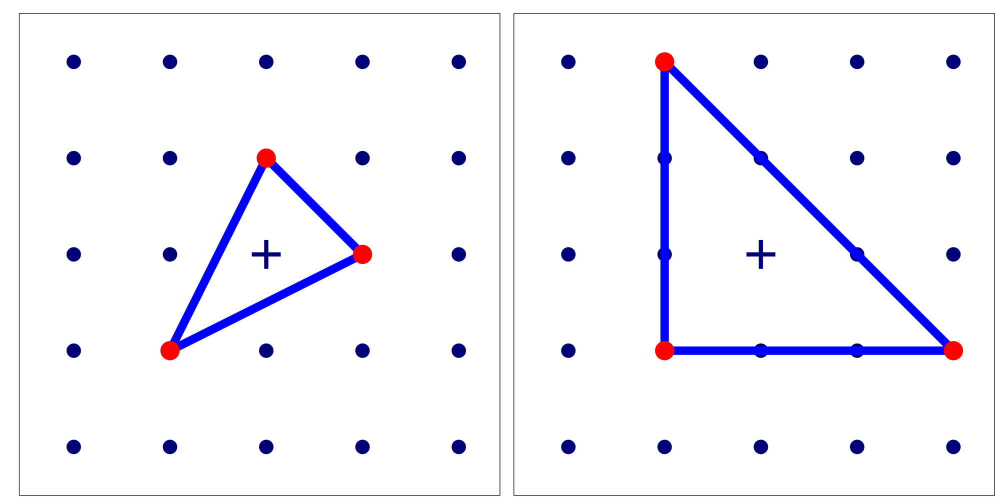

fig. 1 2D reflexive simplex and its polar dual

Dual Varieties over Finite Fields

Reflexive polytopes can be used to describe Calabi-Yau varieties. To construct a pair of varieties from dual reflexive polytopes we take the anticanonical hypersurfaces in the associated toric varieties. The relationship between these pairs of varieties exhibits some of the same properties observed in the mirror symmetry of string theory.

It has been proven for all reflexive simplices that these dual corresponding varieties have an equivalent number of points mod q over the same finite field $\mathbb{F}_q$. Indeed, in higher dimensions we cannot expect an equal number of points on both sides but, rather, a *strong arithmetic mirror symmetry*, meaning the stated equivalent number of solutions over a finite field.

The dual varieties obtained from the the reflexive simplices in fig. 1 are:

$$\mathbb{V}\left(z_1^3+z_2^3+z_3^3+tz_1z_2z_3\right)$$

and

$$\mathbb{V}\left(z_3^3z_4^2z_5z_6^2z_8+z_2^3z_4z_5^2z_7^2z_9+z_1^3z_6z_7z_8^2z_9^2+tz_1z_2z_3z_4z_5z_6z_7z_8z_9\right)$$

where t is a parameter that is varied through our field.

When we count solutions of these dual varieties over various finite fields we discover that the solution count is, in fact, equivalent for all fields tested!

Elliptic Curves

An *isogeny* between two elliptic curves E_1 and E_2 is a non-constant map $\varphi\colon E_1 \to E_2$ that preserves the point addition defined on elliptic curves. These isogenies always exist in dual pairs. If we use the vertices of a reflexive polytope to define an elliptic curve E, we can find a map φ to the points of all curves it is isogenous with.

In the case of our polytopes in fig. 1, we respectively obtain the equivalence isogeny relationships of:

$$(z_1,z_2,z_3) \sim (\alpha^2\lambda z_1, \alpha\lambda z_2, \lambda z_3)$$

and

$$(z_1,z_2,z_3) \sim (\lambda z_1, \lambda z_2, \lambda z_3)$$

By *Tate's Isogeny Theorem*, these similar equivalency relationships (differing only by a parameter α, where $\alpha^3 = 1$) show us that there will indeed always be an equivalent number of solutions to the shown dual varieties over a finite field.

We calculate these equivalency relationships for all 16 classes of 2D reflexive polytopes, and show that the number of points on dual varieties obtained from all reflexive triangles and one pair of reflexive quadrilaterals over the same finite field are equivalent. Four classes of 2D reflexive polytopes are self-dual and admit a single variety. Of particular interest to us, at this point, is the single pair of reflexive quadrilaterals that exhibit this relationship because the 3D cube and octahedron dual pair do not share the same properties.

Picard-Fuchs Equations

Another way we can examine elliptic curves is through *Picard-Fuchs equations*. A Picard-Fuchs equation is an ordinary differential equation that describes how the value of a *period* of a family of varieties changes as we move through the family, where a period is the integral of a *differential form* with respect to a specified subspace. Picard-Fuchs equations encode variations of complex structure in the family.

The *Griffiths-Dwork technique* is a standard, albeit tedious, method for calculating the Picard-Fuchs equation for a given family. We calculate the Picard-Fuchs equation of our variety $V = \mathbb{V}\left(z_1^3+z_2^3+z_3^3+tz_1z_2z_3\right)$ by first recognizing that V is an elliptic curve so it must have a single *holomorphic 1-form* $\in H^{1,0}$. Let $\Omega \in H^{1,0}$, then $\mathcal{P} = \int \mathrm{Res}\left(\frac{\Omega_0}{Q}\right)$ is a period of the holomorphic 1-form. We differentiate $\mathcal{P}$ with respect to t a couple times to get:

$$P' = \int \mathrm{Res}\left(-\frac{\Omega_0}{Q^2}z_1z_2z_3\right) = \int \mathrm{Res}\left(\frac{\Omega_0}{Q^2}\left(-\frac{\mathrm{d}Q}{\mathrm{d}t}\right)\right)$$

$$P'' = \int \mathrm{Res}\left(\frac{\Omega_0}{Q^3}2z_1^2z_2^2z_3^2\right) = \int \mathrm{Res}\left(\frac{\Omega_0}{Q^3}2\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2\right)$$

To find a linear combination of $\mathcal{P}, \mathcal{P}', \mathcal{P}''$ in terms of our parameter t that equals 0, we first need to use the formula:

$$\frac{\Omega_0}{Q^{k+1}}\sum_{i=0}^{n} A_i\frac{\partial Q}{\partial x_i} = \frac{1}{k}\frac{\Omega_0}{Q^k}\sum_{1=0}^{n}\frac{\partial A_i}{\partial x_i} + \mathrm{d}\left(\cdots\right)$$

to reduce the order the pole until it is expressed in terms of lower derivatives. This part is done quickly in Magma mathematics software. Finally, we are left with solving:

$$\begin{bmatrix} 1 & -\frac{1}{t} & \frac{2t}{t^3+27} \\ 0 & \frac{3z_3^3}{t} & -\frac{9tz_3^3}{t^3+27} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To get our Picard-Fuchs equation:

$$t\mathcal{P} + 3t^2\mathcal{P}' + \left(t^3+27\right)\mathcal{P}'' = 0$$

When we calculate the Picard-Fuchs equations for all our 2D reflexive polytopes, we notice that those polytopes that exhibit a strong arithmetic mirror symmetry also have the same Picard-Fuchs equation.

We conjecture that all dual pairs of 3D reflexive simplices will have the same Picard-Fuchs equations, but that the cube and octahedron dual pair will not.

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