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*Report No. 66*

# Sequential Methods in Statistical Process Monitoring
## *Chapter 3: Design of CUSUM Charts*

George Box and José Ramírez

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This report is chapter 3 of a thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Statistics) at the University of Wisconsin–Madison (1989). Thesis adviser: George Box.

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***Report Series***
*in Quality and Productivity*

# Sequential Methods in Statistical Process Monitoring

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Productivity Improvement

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*ABSTRACT*

In the course of improving quality and productivity data is collected with the aim of providing information about processes' performance. Graphical control procedures like the CUSUM chart are a valuable tool since they help us to discover when things might have gone wrong. In this report we show how to design CUSUM charts to monitor variability. Also, it is shown how the CUSUM can be used to detect outliers.

*KEYWORDS: ARL, CUSUM, common causes, nomogram, special causes, SPC variation*

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# CHAPTER 3

# DESIGN OF CUSUM CHARTS

In in the course of improving quality and productivity data is collected with the aim of providing information about processes' performance. With this information decisions are to be made regarding the state of such processes. Graphical control procedures like the cusum chart are a valuable tool since they help us to discover when things might have gone wrong.

When designing cusum charts for monitoring the mean or the process variability, the first question the user might ask is : "what should the parameters of the chart be?". As we have seen, these parameters will depend on the performance,based on the average run length values, he or she expects from the scheme. In this chapter we will show how the contour nomogram provides a simple and flexible way of choosing these parameters.

## 3.1 Monitoring Increases in Process Variability

Let $x_i$ be the outcomes of a process whose variability we wish to monitor; and let $\sigma_a$ and $\sigma_r$ denote the acceptable and rejectable variability levels respectively. A cusum chart for monitoring increases in the variability of the process can be constructed as follows:

1) Determine the acceptable quality level $\sigma_a$, and a set of rejectable quality levels of the process variability. Let $\sigma_r$ denote the smallest of these rejectable values that is considered important enough to be detected quickly.

2) Once the values $\sigma_a$ and $\sigma_r$ have been defined, the reference value $s^2$ is obtained by means of the equation

$$s^2 = \frac{log\left(\frac{\sigma_r^2}{\sigma_a^2}\right)}{\frac{1}{\sigma_a^2} - \frac{1}{\sigma_r^2}}, \tag{3.1}$$

This value can be considered as a "mean" value between the acceptable and rejectable quality levels of the process variability, that provides maximum discrimination between $\sigma_a$ and $\sigma_r$.

3) The value of the decision interval $h$ is chosen with the help of the nomogram to yield the desired ARL values at $\sigma_a$ and $\sigma_r$. A horizontal line is drawn in the nomogram, corresponding to the value of $\sigma_r/\sigma_a$, giving various possible combinations of $h$, $L_a$ and $L_r$.

4) For each observation $x_k$ compute the quantities

$$\begin{aligned} r_k &= [(x_k - \mu)^2 - s^2], \\ CS_k^+ &= max(0, CS_{k-1}^+ + r_k) \qquad \text{where} \quad CS_0^+ = 0. \end{aligned} \tag{3.2}$$

Here *max* means that at every time $k$, starting at $CS_0^+ = 0$, we will take the maximum between 0 and the cumulative sum $CS_{k-1}^+ + r_k$. In other words, we will only plot the sums whenever they are relevant towards taking a decision; i.e., when they are positive, and will reset to zero whenever the sum is negative.

5) Plot, or record, the value of the cumulative sums $CS_k^+$ until they either exceed the decision interval $h$, or reach zero again.

6) An increase in process variability is indicated whenever $CS_k^+ \geq h$.

Two numerical examples will help to clarify how to design cumulative sum charts using the contour nomogram.

*Example 1:* Suppose we wish to design a cusum chart to monitor changes in the variability of a process that has an acceptable quality level of $\sigma_a = 2$; and that the smallest change we want to detect is given by $\sigma_r = 4$. With these two values a reference value $s^2$ is calculated by using formula 3.1, giving $s^2 = 7.39$. A horizontal line can now be drawn on the contour nomogram at the point $\sigma_r / \sigma_a = 2$, giving various possible combinations for $L_a$ and $L_r$.

Figure 3.1 shows the contour nomogram with the line at $\sigma_r / \sigma_a = 2$, and table 3.1 summarizes the possible charts that can be designed by using integer values of the decision interval $h$.

Figures 3.2a and 3.2b, show 30 observations from a $N(0,4)$ for which the variance changed to $\sigma_r^2 = 16$ after observation 20. Figure 3.2a shows a cusum chart for these data with decision interval $h = 40$ and average run length values $L_a = 552$ and $L_r = 6.7$. This means that on the average we expect to have false positives approximately every 552 observations and to detect changes 7 observations after the change has occurred. Note that a lack of control, increase in variability, is indicated after observation 26. The cusum chart with parameters $s^2 = 7.39$, $h = 72$, $L_a = 11398$ and $L_r = 10.5$, figure 3.2b, allows us to detect the change after observation 28.

| $s^2 = 7.39$ | | | |
|---|---|---|---|
| $\frac{h}{\sigma_a^2}$ | $h$ | $L_a$ | $L_r$ |
| 5 | 20 | 73 | 4.7 |
| 6 | 24 | 112 | 4.8 |
| 7 | 28 | 169 | 5.3 |
| 8 | 32 | 252 | 5.8 |
| 9 | 36 | 374 | 6.2 |
| 10 | 40 | 552 | 6.7 |
| 11 | 44 | 812 | 7.2 |
| 12 | 48 | 1190 | 7.7 |
| 13 | 52 | 1742 | 8.1 |
| 14 | 56 | 2543 | 8.6 |
| 15 | 60 | 3710 | 9.1 |
| 16 | 64 | 5385 | 9.5 |
| 17 | 68 | 7844 | 10.0 |
| 18 | 72 | 11398 | 10.5 |
| 19 | 76 | 16531 | 10.9 |
| 20 | 80 | 23884 | 11.4 |

**Table 3.1** Values of $h$, $L_a$ and $L_r$

In some instances it is possible to specify the values of $L_a$ and $L_r$ that are to be used in order to detect an increase in variability from some known acceptable level $\sigma_a$. In this case the nomogram can be used not only for designing the cusum

**Figure 3.1** Contour Nomogram with line at $\sigma_r / \sigma_a = 2$, giving various combinations for the action limit $h$, and the average run lengths $L_a$ and $L_r$.

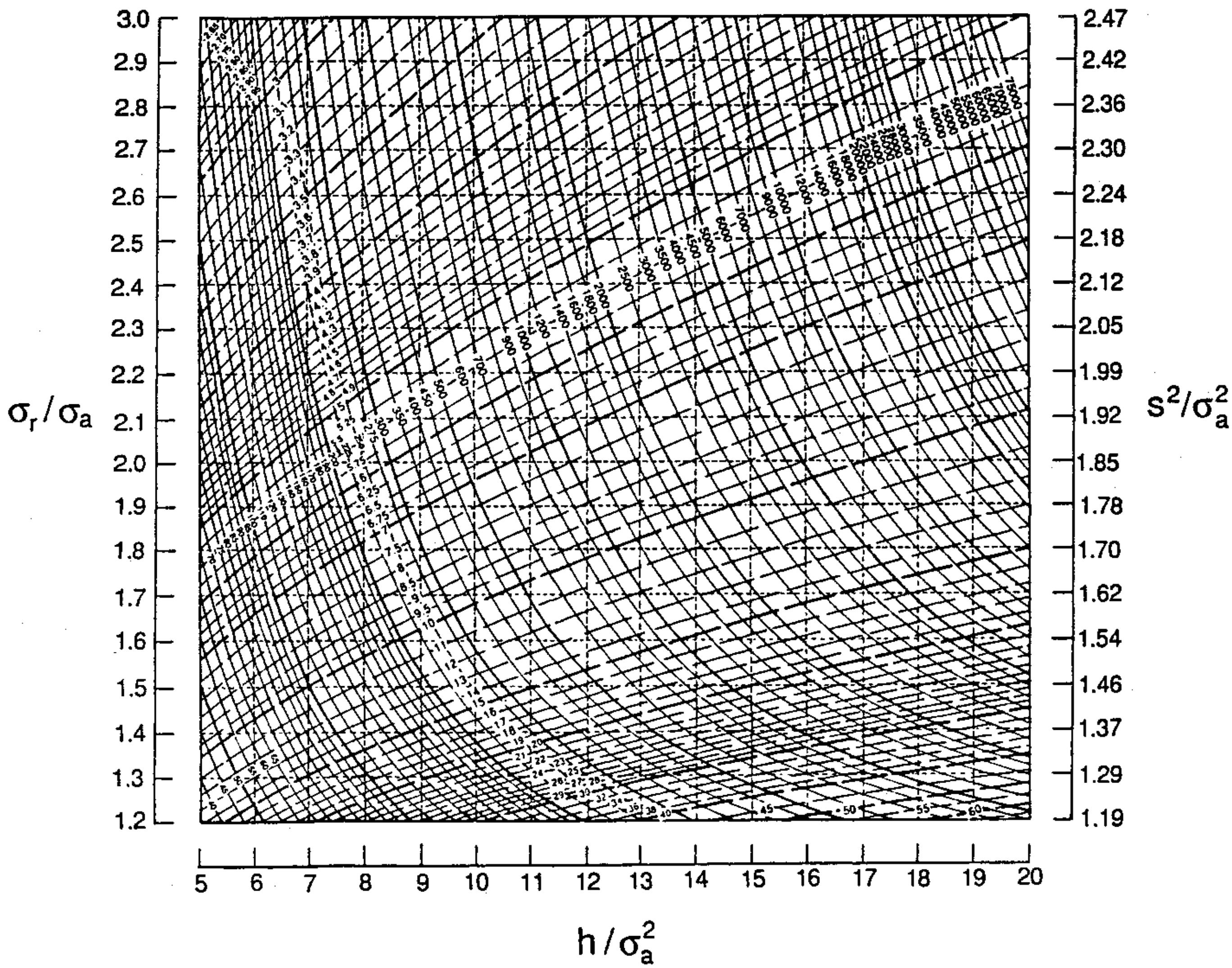


**Figure 3.2a** CUSUM chart with parameters $h = 40$ and $s^2 = 7.39$, for the outcomes of a process that are distributed as $N(0,4)$. The variance changed to 16 after observation 20.

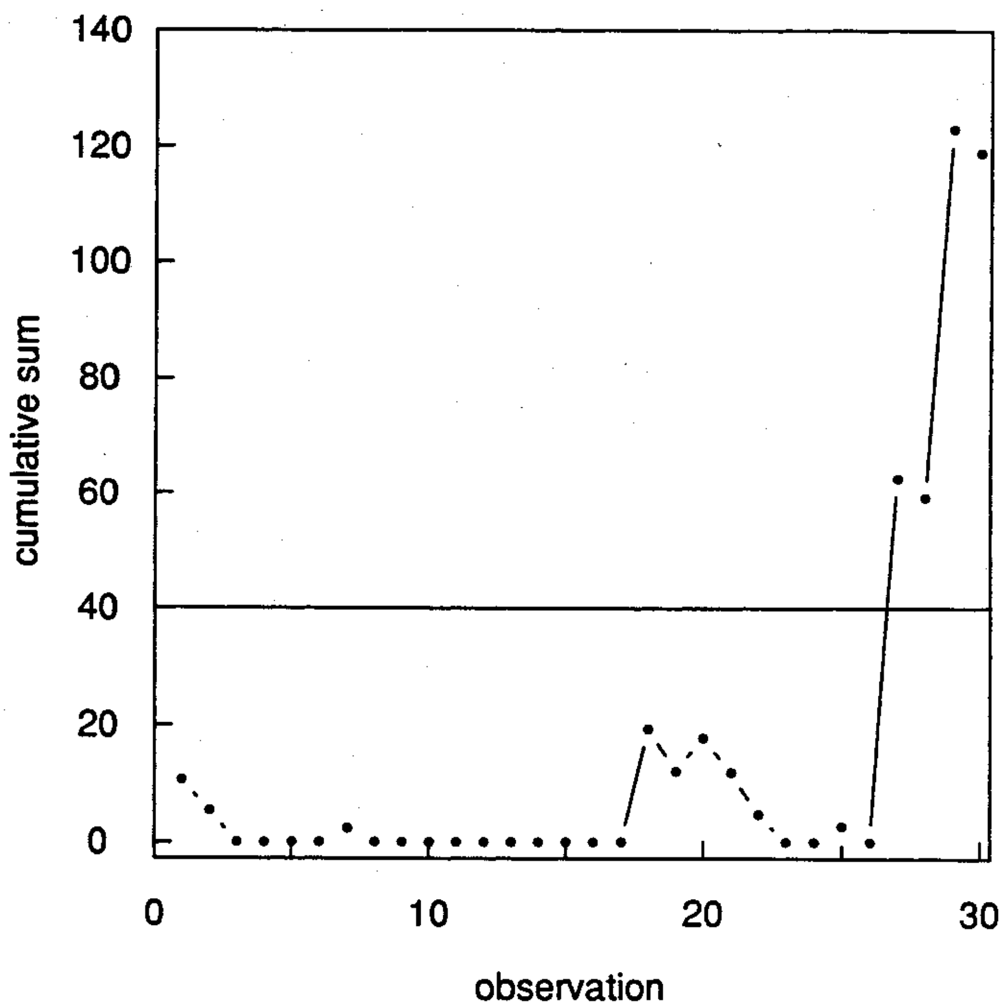


**Figure 3.2b** CUSUM chart with parameters $h = 72$ and $s^2 = 7.39$, for the outcomes of a process that are distributed as $N(0,4)$. The variance changed to 16 after observation 20.

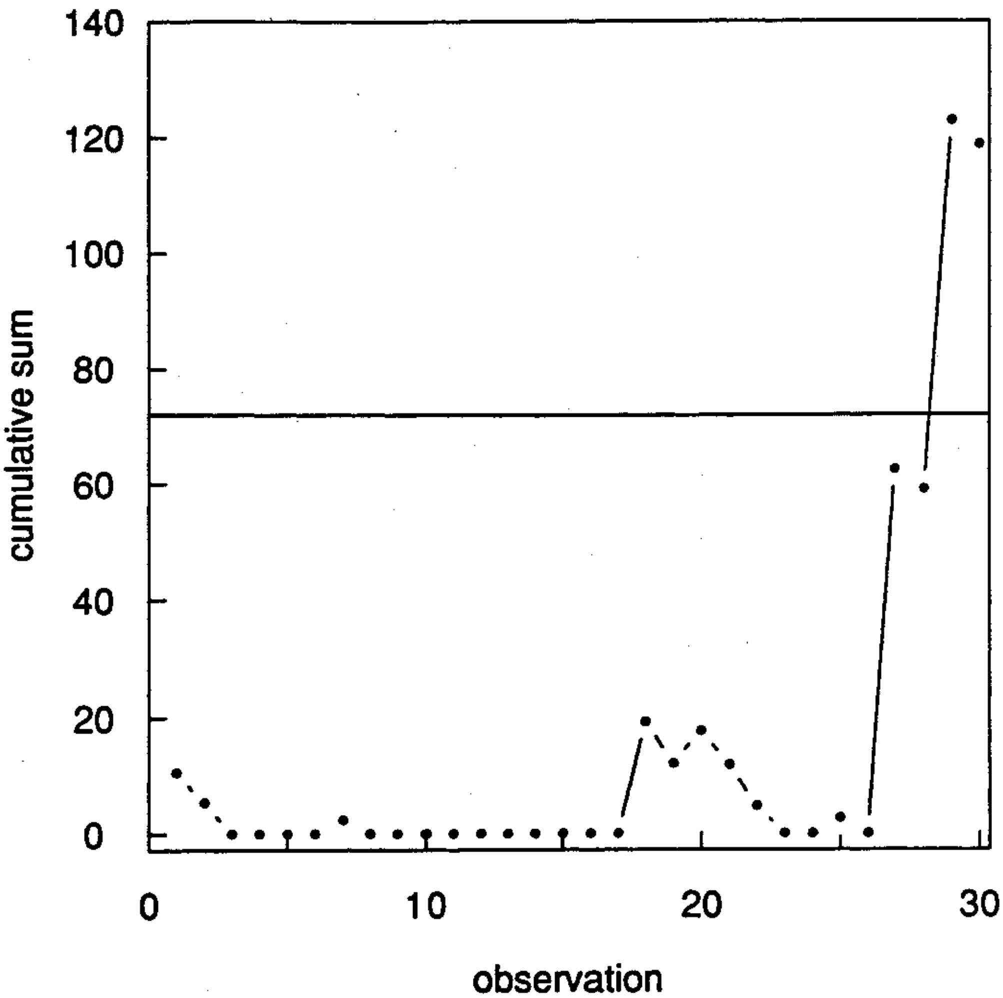


chart but to describe the magnitude of the change that we can detect.

*Example 2:* Suppose we want to design a cusum chart with approximate average run lengths, $L_a = 1200$ and $L_r = 7$ for a process with mean centered at target and equal to 13 and variability $\sigma_a = 2$. From the contour nomogram the point of intersection of the contours $L_a = 1200$ and $L_r = 7$ gives $\sigma_r / \sigma_a \approx 2.08$ and $h / \sigma_a^2 = 11.7$ as illustrated in figure 3.3. In other words, with $L_a$, $L_r$ and $h$ as above the chart would be able to detect changes in the standard deviation of at least 108%. To obtain $s^2$ we use formula (3.1) with $\sigma_r = 2 \times 2.08 = 4.16$, which gives $s^2 = 7.62$. The decision interval is equal to $h = 4 \times 11.7 = 46.8$.

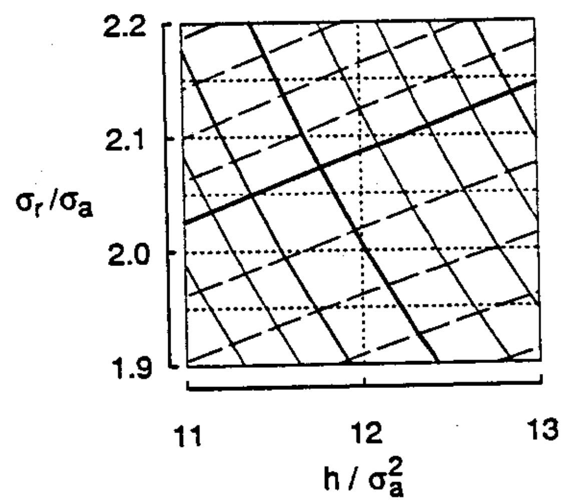



**Figure 3.3** Section of contour nomogram showing the contours of $L_a = 1200$ and $L_r = 7$.

The procedure consists of calculating at each point $k$ the quantity $r_k = [(x_k - 13)^2 - 7.62]$, and plotting $CS_k^+ = max(0, CS_{k-1}^+ + r_k)$. A lack control is indicated whenever $CS_k^+ \geq 46.8$.

### 3.2 Monitoring Decreases in Process Variability

So far we have been concerned with monitoring increases in the variability of the process; and although it is true that we do not want the variability to increase, our main goal should be to continuously reduce it, so to minimize loss and to obtain better and more reliable products. For consistency with the previous sections we have kept the notation $\sigma_a$ and $\sigma_r$ to denote the old and new level of the process variability respectively. Here the subscript $r$ should be viewed as a desirable level rather than a rejectable level.

We now focus our attention in the design of cumulative sum charts for monitoring decreases in the variability of the process from a level $\sigma_a$ to a level $\sigma_r$. In this case we can write $\sigma_a = m\sigma_r$, where $m > 1$. The design of these charts is simplified by noting that the reference value $s_d^2$ for a decrease in variance by an amount of $1/m$, can be written in terms of the reference value $s_i^2$ of an increase in variance from 1 to $m$. In other words

$$s_d^2 = \frac{log\,(\sigma_r^2/\sigma_a^2)}{\dfrac{1}{\sigma_a^2} - \dfrac{1}{\sigma_r^2}} = \sigma_r^2 \frac{log\,(1/m^2)}{\dfrac{1}{m^2} - 1} = \sigma_r^2 s_i^2 \tag{3.3}$$

The decision interval $h_d$ satisfies a similar relation; i.e., $h_d = -\sigma_r^2 h_i$; where $h_i$ is the decision interval for a chart designed to monitor increases in variances from $\sigma_a = 1$ to $\sigma_r = m$. Note that, as in the case for cumulative sum charts for monitoring the mean, the decision interval $h_d$ is the negative of $h_i$ but adjusted by the variability we want to detect. In general however, the action limits $h_i$ and $h_d$ will not be symmetric.

The design of cumulative sum charts for monitoring decreases in process variability is similar to the design of cumulative sum charts for monitoring increases in process variability given in section 3.1 with the following modifications:

1) Let $\sigma_a$ denote the current process variability and let $\sigma_r$ denote the smallest decrease in process variance, as a fraction of $\sigma_a$ that we want to be able to detect; i.e., $\sigma_r/\sigma_a = 1/m$.

2) Compute the reference value $s_d^2$ using formula (3.1) with the values of $\sigma_a$ and $\sigma_r$ obtained in (1).

3) Draw a horizontal line in the contour nomogram at the value $\sigma_a/\sigma_r = m$. Choose the decision interval $h/\sigma_a^2$ that yields the desired ARL values at $L_a$ and $L_r$. The decision interval for the chart is given by $h_d = -h/m^2$.

4) For each observation $x_k$ compute the quantities

$$
\begin{aligned}
r_k &= [(x_k - \mu)^2 - s_d^2], \\
CS_k^- &= min(0, CS_{k-1}^- + r_k) \qquad \text{where} \quad CS_0^- = 0.
\end{aligned}
\tag{3.4}
$$

Here *min* means that at every time $k$, starting at $CS_0^+ = 0$, we will take the minimum between 0 and the cumulative sum $CS_{k-1}^- + r_k$. In other words, we will only plot the sums whenever they are relevant towards taking a decision; i.e., when they are negative, and will reset to zero whenever the sum is positive.

5) Plot, or record, the value of the cumulative sums $CS_k^-$ until they either exceed the decision interval $h_d$, or reach zero again.

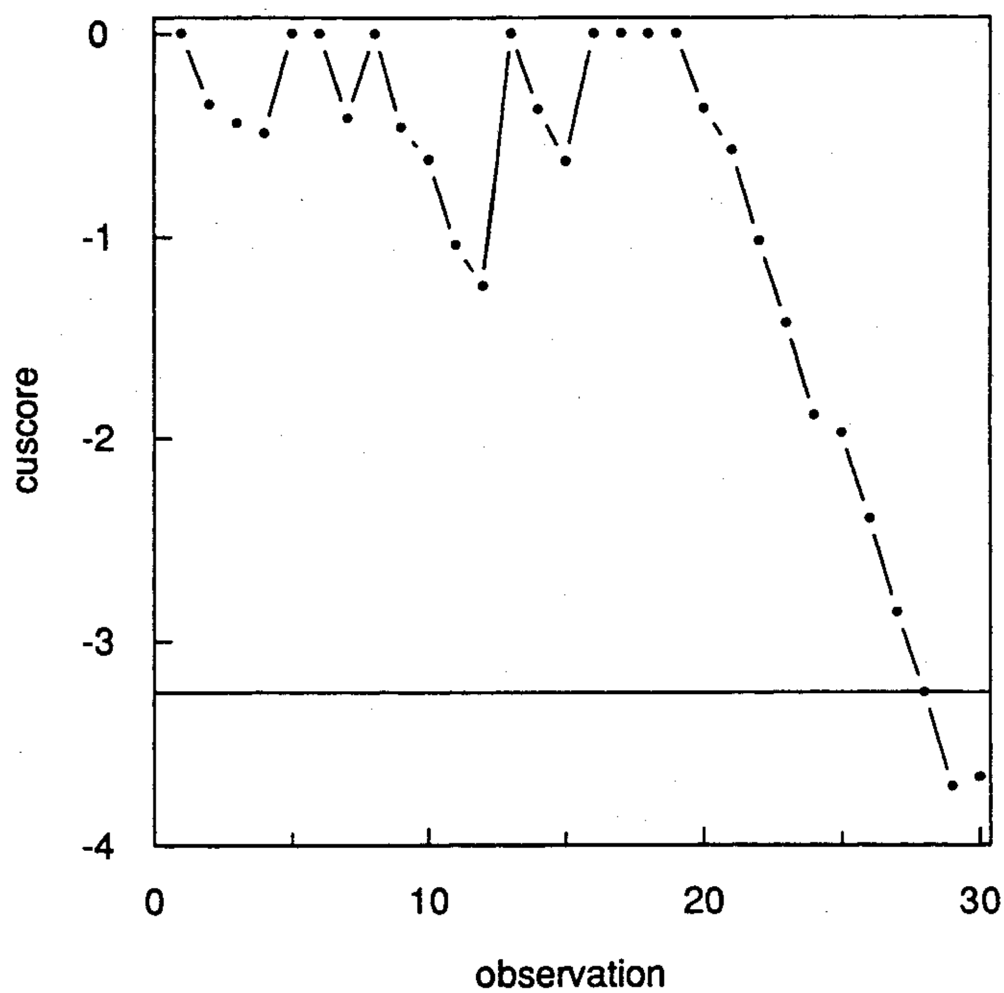
6) A lack of control, increase in process variability, is indicated whenever $CS_k^- \leq h_d$.

The reference value $s_d^2$ is then computed by the same formula used in the design of charts for monitoring increases in variabilty. The action limit $h_d$ is given $h_d = -\sigma_r^2 h_i$, where $h_i$ is obtained using the contour nomogram at the point $\sigma_r / \sigma_a = m$ to yield the desired average run lengths.

As an example suppose that we have a process with target value $\mu = 0$ and we want to monitor a 50% decrease in variability. In this case $\sigma_r / \sigma_a = ½$ and using formula 3.3, we get $s_d^2 = 0.46$. By drawing a line in the contour nomogram, at the point $\sigma_r / \sigma_a = 2$, we see that a decision interval $h_i / \sigma_a^2 = 13$ gives average run length values $L_a = 1600$ and $L_r = 8$. The decision interval $h_d$ for the chart is then $h_d = -13/4 = -3.25$. At each point $k$ we compute $r_k = [x_i^2 - 0.46]$ and plot the cumulative sum $CS_k^- = min(0, CS_{k-1}^- + r_k)$.

Figure 3.4 shows 30 observations from a $N(0,1)$, for which the variance changed to 1/4 after observation 20, and the CUSUM chart with parameters $s_d^2 = 0.46$ and $h_d = -3.25$, as described above. Note the change of behaviour of the chart after observation 20 and that a lack of control is indicated at observation 28.

**Figure 3.4** CUSUM Chart with parameters $h_d = -3.25$ and $s_i^2 = 0.46$, for the outcomes of a process that are distributed as $N(0,1)$. The variance changed to 1/4 after observation 20.



## 3.3 Subgroups of Size Greater than One

In some cases, especially when running Shewhart control charts, data is collected in subgroups that have size greater than one. The procedures described in chapter 2 and in sections 3.1 and 3.3 can still be used for the design of CUSUM charts.

When the sample size at each time $k$ is greater than 1, $m$ say, we can compute the mean of the $m$ observations at time $k$, $\overline{X}_{k\cdot} = \frac{1}{m}\sum_{j=1}^{m} x_{kj}$, and use the $\overline{X}_{k\cdot}$ as observations in the CUSUM chart. In other words, at each time $k$ we compute

$$r_k = [(\overline{X}_{k\cdot} - \mu)^2 - s_j^2], \qquad j=i,d, \tag{3.5}$$

and calculate $CS_k^+$ or $CS_k^-$ depending on what type of change we want to monitor.

The contour nomogram of chapter 2 can be used to select the decision interval and the corresponding average run lengths $L_a$ and $L_r$. If the observations have variances $\sigma_a^2$ and $\sigma_r^2$, at the acceptable and rejectable quality level respectively, $s^2$ is the reference value obtained by using $\sigma_a^2$ and $\sigma_r^2$ and $h/\sigma_a^2$ is the decision interval obtained from the contour nomogram, the reference value and decision interval, $s_m^2$ and $h_m^2$, for subgroups of size $m$ are given by

$$s_m^2 = \frac{1}{m}s^2 \quad \text{and} \quad h_m = \frac{h}{m}. \tag{3.6}$$

This follows because $\overline{X}_{k\cdot}$ has variance $\sigma_a^2/m$ or $\sigma_r^2/m$ at the acceptable or rejectable quality level respectively.

### 3.4 Unknown Mean

In previous sections we have assumed that the process mean was known. There are some situations in which we do not know the process mean but we have an estimate based on past data. We can use this estimate of the mean to compute the cumulative sums by computing at each time $k$

$$r_k = [(x_i - \overline{X})^2 - s^2] \tag{3.7}$$

and the corresponding $CS_k^+$ or $CS_k^-$ for monitoring increases or decreases respectively.

The result of using $\overline{X}$ instead of $\mu$ is that of increasing the average sample number $N(0)$ (equation 2.7), by one unit, see for example Wald (1947) or Girshick (1946). Hence the new average run length values, here denoted by $L'$ become

$$L'_j = \frac{N(0)+1}{1-P(0)} = L_j + \frac{1}{1-P(0)} \qquad j=a,r \,. \tag{3.8}$$

Showing that the ARL is larger by a factor of $1/(1-P(0))$.

As $\sigma_r$ gets large, the probability of returning to zero, $P(0)$, tends to 0, and hence for large values of $\sigma_r$ $L'_r \simeq L_r$. In other words, we can use the $L_r$ values given by the contour nomogram when monitoring large shifts in process variance. On the other hand at the acceptable quality level $\sigma_a$, the probability of returning to zero tends to 1 and from equation 3.8 we see that $L'_a \gg L_a$; which indicates that the $L_a$ values given by the nomogram are conservative when monitoring small shifts in process variance.

By using the values of $L$ and $P(0)$ obtained by solving the integral equations over the grid $5 \leq h \leq 20$ and $1.2 \leq (\sigma_r/\sigma_a) \leq 3$, we can compute approximate formulas for $L'$ as a function of $L$. These formulas are given by

$$\begin{aligned} L'_a &\approx 1.82 \times L_a \\ L'_r &\approx 1.22 \times L_r \end{aligned} \tag{3.9}$$

As an example the data in table 3.2 (from Johnson and Leone 1977), give the coded results for 20 samples of size 4 of the measurements of elongation of polyethylene test specimens, the averages $\overline{X}_{k\cdot}$, ranges and values of the cumulative sums $CS_k^+$. Here the target value is the overall average $\overline{x} = 8.56$.

Suppose that we are interested in monitoring increases in variance of about 30% ($\sigma_r/\sigma_a = 1.3$). Using the first 40 measurements to obtain an estimate of the process variability we get $\sigma_a = 4.8$ and $\sigma_r = 6.24$. This gives, for subgroups of size 1, a reference value $s^2 = 29.61$, and a decision interval $h/\sigma_a^2 = 6$ yielding ARL values $L_a = 55$ and $L_r = 13$. For subgroups of size 4 equation 3.6 gives $s^2 = 29.61/4 = 7.40$ and $h = 23.02 \times 6/4 = 34.5$. Using the approximate formulas 3.9 we obtained the ARL values $L'_a \approx 100$ and $L'_r \approx 16$.

The cumulative sums $CS_k^+ = max(0, CS_{k-1}^+ + r_k)$ are computed for each sample $k$, where $r_k = [(\overline{X}_{k\cdot} - 8.56)^2 - 7.40]$, and compared with the decision interval 34.5.

Figure 3.5a and 3.5b show the Shewhart $R$ chart with action limit UCL=$2.28 \times \overline{R} = 2.28 \times 9.1 = 20.77$, as well as the CUSUM chart. Note that the $R$

**Figure 3.5a** R-Chart for the measurements of elongation of polyethylene given in table 3.2.

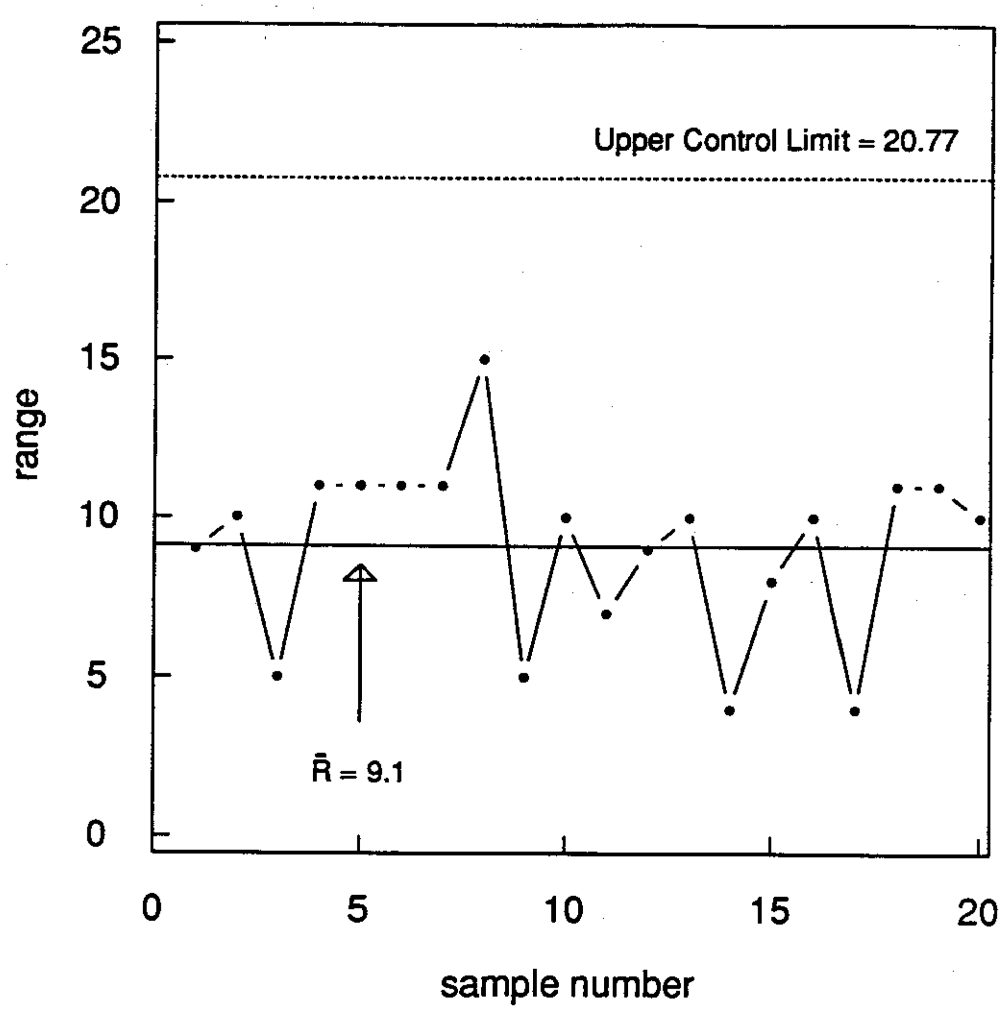
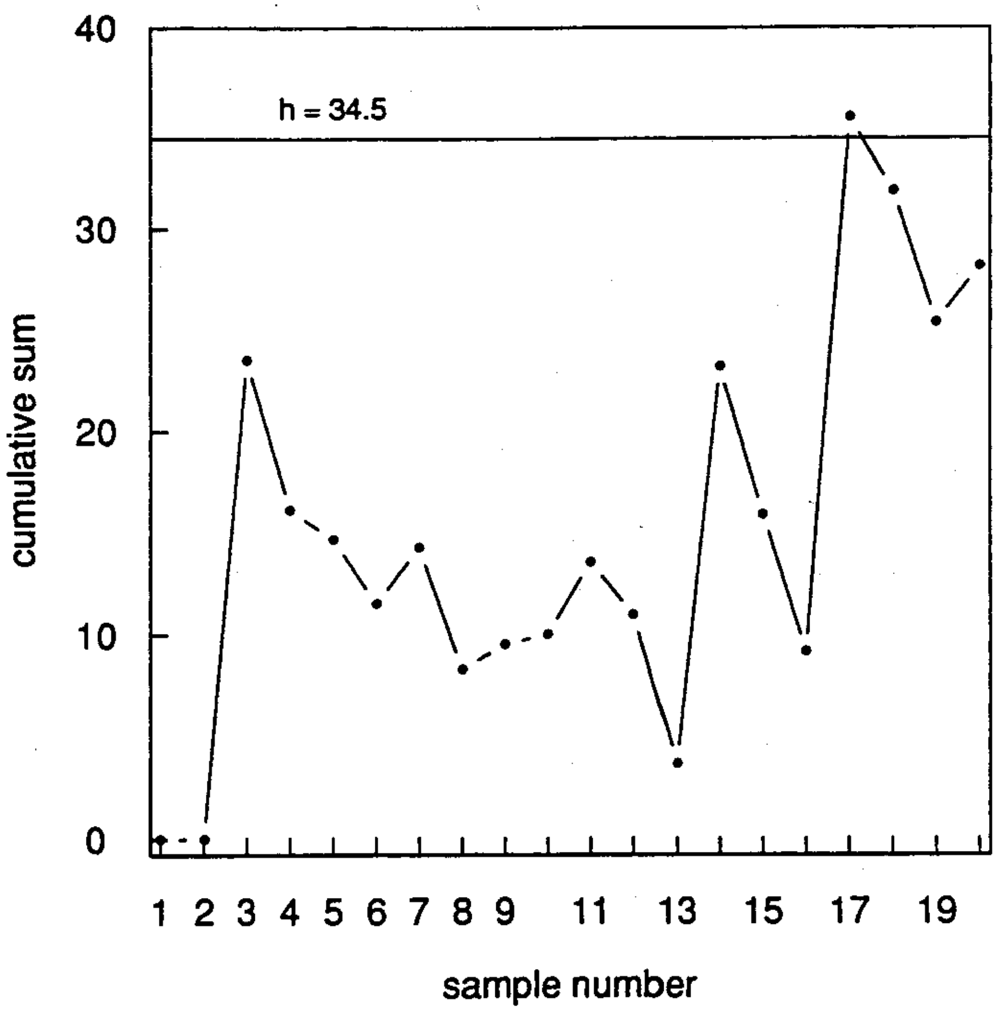


**Figure 3.5b** Cusum chart for the measurements of elongation of polyethylene given in table 3.2. A lack of control is indicated at sample 17.



| sample number | measurements | | | | average | range | $CS_k^+$ |
|---|---|---|---|---|---|---|---|
| 1 | 12 | 3 | 10 | 7 | 8.00 | 9 | 0.0 |
| 2 | 11 | 10 | 1 | 5 | 6.75 | 10 | 0.0 |
| 3 | 2 | 6 | 3 | 1 | 3.00 | 5 | 23.51 |
| 4 | 13 | 8 | 2 | 12 | 8.75 | 11 | 16.14 |
| 5 | 9 | 14 | 5 | 16 | 11.00 | 11 | 14.69 |
| 6 | 11 | 2 | 12 | 1 | 6.50 | 11 | 11.54 |
| 7 | 12 | 5 | 16 | 14 | 11.75 | 11 | 14.31 |
| 8 | 10 | 1 | 12 | 16 | 9.75 | 15 | 8.32 |
| 9 | 13 | 10 | 9 | 14 | 11.50 | 5 | 9.56 |
| 10 | 5 | 1 | 11 | 6 | 5.75 | 10 | 10.06 |
| 11 | 8 | 5 | 7 | 1 | 5.25 | 7 | 13.61 |
| 12 | 6 | 10 | 12 | 15 | 10.75 | 9 | 11.0 |
| 13 | 15 | 5 | 6 | 7 | 8.25 | 10 | 3.70 |
| 14 | 13 | 16 | 12 | 14 | 13.75 | 4 | 23.23 |
| 15 | 6 | 13 | 5 | 9 | 8.25 | 8 | 15.92 |
| 16 | 3 | 4 | 11 | 13 | 7.75 | 10 | 9.18 |
| 17 | 2 | 3 | 5 | 1 | 2.75 | 4 | 35.53 |
| 18 | 14 | 5 | 7 | 16 | 10.50 | 11 | 31.89 |
| 19 | 16 | 5 | 11 | 6 | 9.50 | 11 | 25.37 |
| 20 | 6 | 15 | 16 | 10 | 11.75 | 10 | 28.14 |

**Table 3.2** Coded results of measurements of elongation of polyethylene.

chart shows no sign of out control while the CUSUM chart indicates a lack of control at sample 17. This shows that the cumulative sum chart is very good at detecting specific departures from the acceptable levels, e.g. a 30% increase. However, it is not always the case that we know exactly the amount of change we want to detect. In this case, the Shewhart chart continues to be an invaluable tool.

The use of cumulative sum charts in general, is particularly effective when we know the process under study is stable; i.e. its performance is predictable, and we want to monitor closely any departures, by a specific amount, from the acceptable quality levels.

### 3.5 Approximations Formulas for the Average Run Length

We have seen that the contour nomogram is an useful aid in the design of cusum charts since it allows us to choose the parameters for the chart based on some predetermined average run length values $L_a$ and $L_r$. It will be nice, however, to have a formula that enable us to compute an approximate A.R.L value in terms of the decision interval $h$ and the reference value $s^2$.

For a sequential test with boundaries $h_0$ and $h_1$, approximation formulas (Wald 1947), for the operating characteristic $P(h)$ and the average sample number $N(h)$ are as follows (see also Barnard (1946) equation 14),

$$\begin{aligned} P(h_0) &= \frac{e^{-th_1} - 1}{e^{-th_1} - e^{-th_0}}, \\ N(h_0) &= \frac{P(h_0)\, h_0 + [1 - P(h_0)]\, h_1}{\sigma^2 - s^2}. \end{aligned} \tag{3.10}$$

Where $t$ is such that $\sigma^2 = \dfrac{e^{2ts^2} - 1}{2t}$.

The lower and upper boundaries for a cusum chart are respectively, $h_0 = 0$ and $h_1 = h$. Recall that (see equation 2.9), the average run length for a test starting at 0, can be written as a function of $P(0)$ and $N(0)$; i.e. $L(0) = \dfrac{N(0)}{1 - P(0)}$. The

Wald approximation formulas therefore, do not apply since for $h_0 = 0$, $P(0) = 1$ and $N(0) = 0$. One way of modifying this procedure, see for example Reynolds (1975), is to let $h_0 = \varepsilon$, for $\varepsilon$ small, and then compute $L(0)$ as the limit when $\varepsilon$ tends to 0 of $L(\varepsilon)$, where $L(\varepsilon) = \frac{N(\varepsilon)}{1 - P(\varepsilon)}$.

$$\begin{aligned} L(\varepsilon) &= \frac{N(\varepsilon)}{1-P(\varepsilon)} = \frac{1}{\sigma^2 - s^2}\left[ h + \frac{\varepsilon P(\varepsilon)}{1-P(\varepsilon)} \right] \\ &= \frac{1}{\sigma^2 - s^2}\left[ h + \frac{\varepsilon (e^{-th} - 1)}{(1 - e^{-t\varepsilon})} \right] . \end{aligned} \tag{3.11}$$

Letting $\varepsilon$ go to 0 we get

$$L(0) = \lim_{\varepsilon \to 0} L(\varepsilon) = \frac{1}{\sigma^2 - s^2}\left[ h + \frac{(e^{-th} - 1)}{t} \right] . \tag{3.12}$$

By letting $\sigma^2$ be equal to the acceptable or rejectable variances, $\sigma_a^2$ and $\sigma_r^2$, we obtain approximations for $L_a$ and $L_r$ respectively.

Since the A.R.L is a function of the decision interval $h$ and the reference value $s^2$, which in turn is a function of $\sigma_r / \sigma_a$, we can try to approximate the A.R.L surface by a second degree polynomial in these two variables. Goldsmith and Whitfield (1961) gave an empirical formula for the A.R.L expressing $loglog(L)$ as a function of the leading distance $d$ and the angle $\theta$ of the V-mask.

Using this approximation we fit the model

$$loglog(L_i) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \beta_4 x_1^2 + \beta_5 x_2^2 + \varepsilon. \tag{3.13}$$

Where $x_1 = log\,(h)$, $x_2 = (\sigma_r / \sigma_a)^{-1}$ and $L_i$, $i = a,r$, denotes the average run length at the acceptable and rejectable quality levels.

The table below shows the estimates and standard errors for the parameter of the model given by equation (3.13) for $L_a$ and $L_r$.

| | $L_a$ | | $L_r$ | |
|---|---|---|---|---|
| Parameter | Estimate | Std. Err. | Estimate | Std. Err. |
| $\beta_0$ | 0.822 | 0.017 | -1.898 | 0.019 |
| $\beta_1$ | 0.401 | 0.012 | 0.474 | 0.014 |
| $\beta_2$ | 0.236 | 0.027 | 4.061 | 0.031 |
| $\beta_3$ | -0.320 | 0.006 | -0.107 | 0.007 |
| $\beta_4$ | 0.078 | 0.002 | -0.012 | 0.003 |
| $\beta_5$ | -0.248 | 0.019 | -1.591 | 0.022 |

**Table 3.3** Parameter estimates for the model in equation 3.13.

The approximation is very good specially for values of $h < 15$. For large values of $h$ and $s^2$ the approximation given by equation (1) tends to underestimate the value of $L_a$. Table 3.4 give the values of $L_a$ and $L_r$ obtained by solving the integral equations and by using equation (1) for different values of $h$ and $s^2$.

**Table 3.4** Values of the average run length at the acceptable, $L_a$, and rejectable, $L_r$, quality levels for the approximation formulas given by equation 3.13.

| Average Run Length $L_a$ | | | | |
|---|---|---|---|---|
| $h$ | $s^2 = 1.46$ | | $s^2 = 1.85$ | |
| | integral equations | eq. 3.13 | integral equations | eq. 3.13 |
| 5 | 49.26 | 51.81 | 73.65 | 76.69 |
| 7 | 99.76 | 101.13 | 169.14 | 175.39 |
| 10 | 260.52 | 258.71 | 552.55 | 565.52 |
| 13 | 638.02 | 631.32 | 1742.42 | 1735.57 |
| 15 | 1139.50 | 1124.53 | 3710.19 | 3601.93 |
| 17 | 2015.03 | 1982.60 | 7844.64 | 7402.12 |
| 20 | 4676.26 | 4573.27 | 23884.25 | 21514.75 |
| **Average Run Length $L_r$** | | | | |
| $h$ | $s^2 = 1.46$ | | $s^2 = 1.85$ | |
| | integral equations | eq. 3.13 | integral equations | eq. 3.13 |
| 5 | 7.60 | 7.79 | 4.36 | 4.32 |
| 7 | 9.96 | 10.15 | 5.31 | 5.26 |
| 10 | 13.62 | 13.82 | 6.72 | 6.65 |
| 13 | 17.34 | 17.71 | 8.12 | 8.03 |
| 15 | 19.85 | 20.43 | 9.06 | 8.95 |
| 17 | 22.39 | 23.25 | 9.99 | 9.89 |
| 20 | 26.19 | 27.69 | 11.38 | 11.31 |

**3.6 Outlying Observations**

In most statistical procedures it is assumed that the observations are generated by some underlying stochastic model. Outlying observations are those observations that appear to be inconsistent with the rest of the sample. They are often seen as contaminants; i.e. observations that come from some other distribution, therefore reducing and distorting the information contained in the data (Grubbs 1969; Barnett and Lewis 1984).

Following the approach of Box and Tiao (1968), when no knowledge is available as to which particular observations are outliers, it is supposed that there was a small prior probability $\alpha$ that any given observation came from a distribution different from the one generating the observations, and a complementary prior probability $(1-\alpha)$ that it came from the distribution generating the observations.

It is supposed that a good observation is normally distributed about its fixed mean $\mu$ with variance $\sigma^2$ and that a bad observation is normally distributed about the same mean but with a larger variance $k^2\sigma^2$. Therefore detecting outlying observations is equivalent to detecting changes in variability from a level $\sigma^2$ to a level $k^2\sigma^2$.

By using the posterior distributions of the observations under the two models a sequential probability ratio test can be constructed that takes into account the prior probabilities and allows us to detect the bad observations.

Let the two models be $N(\mu,\sigma^2)$ and $N(\mu,k^2\sigma^2)$, $k$ fixed, with prior probabilities $\alpha$ and $(1-\alpha)$ respectively. Also assume that $\mu$ and $\log\sigma$ are locally independent and uniform a priori. For an observation $x_i$ we have

$$f(x_i \mid \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]$$

$$f(x_i \mid k^2\sigma^2) = \frac{1}{k\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2k^2\sigma^2}\right] \tag{3.14}$$

$$p(\mu,\sigma) \propto \frac{1}{\sigma}.$$

The likelihood ratio of the posterior distributions that an observation $x_i$ comes from the distribution $N(\mu,\sigma^2)$ or the distribution $N(\mu,k^2\sigma^2)$ is equal to

$$\frac{\alpha \exp\left[-\frac{(x_i-\mu)^2}{2k^2\sigma^2}\right]}{k(1-\alpha)\exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right]} \quad . \tag{3.15}$$

After taking logs, and by letting $\phi = 1 - \frac{1}{k^2}$, the likelihood ratio for a sample of $n$ observations is proportional to

$$\sum_{i=1}^{n}\left[(x_i-\mu)^2 - \frac{2\sigma^2}{\phi}\log\left[\frac{k(1-\alpha)}{\alpha}\right]\right] \tag{3.16}$$

Note that the reference value $s^2$ is now a function of the prior probabilities $\alpha$ and $(1-\alpha)$; i.e.

$$s^2(\alpha) = \frac{2\sigma^2}{\phi}\log\left[\frac{k(1-\alpha)}{\alpha}\right] \tag{3.17}$$

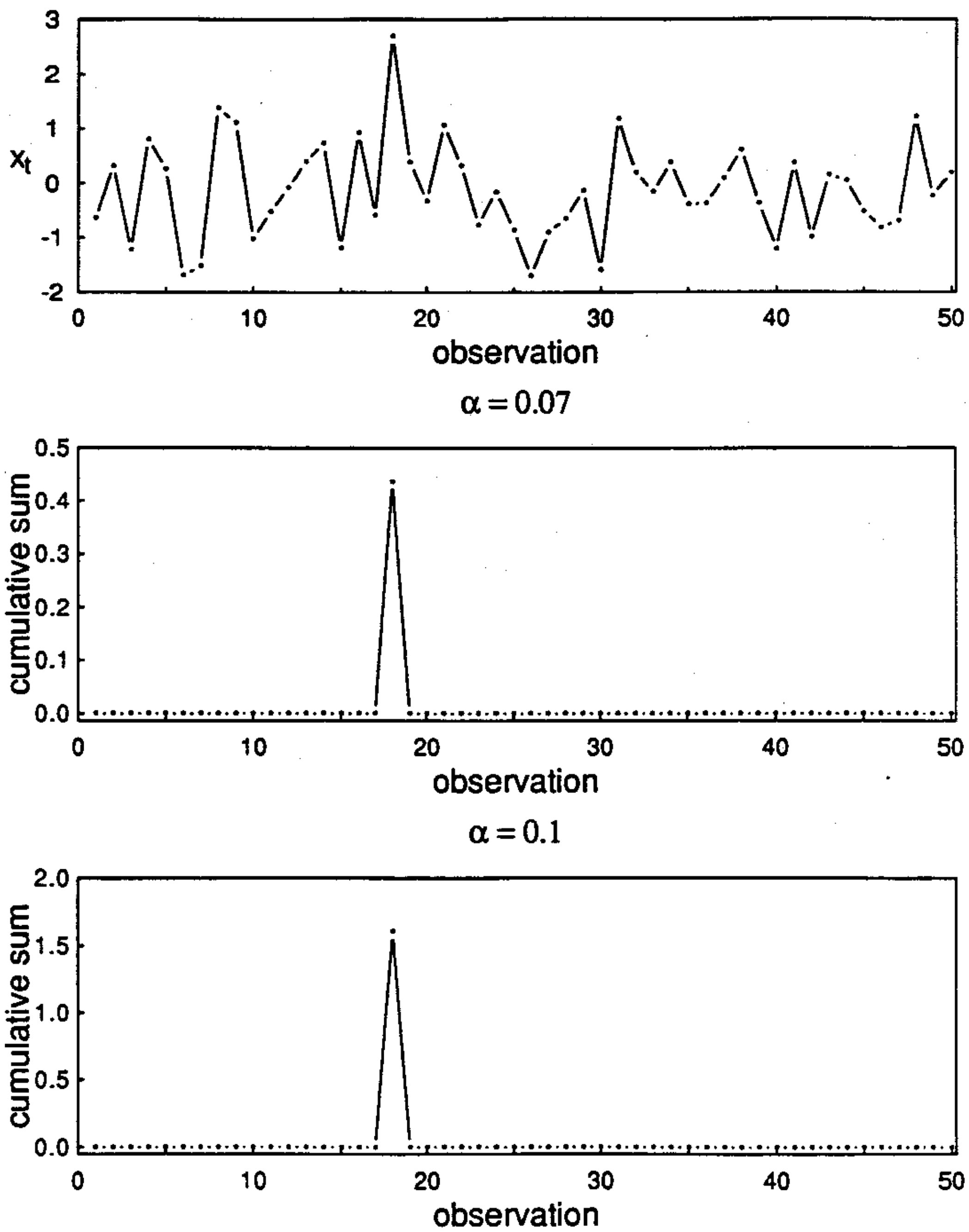
When $\alpha$ tends to 0; i.e. when the prior probability of an observation being an outlier tends to 0, $s^2(\alpha)$ tends to $\infty$, and therefore $CS_n^+$ will always be negative. When $\alpha = ½$, $s^2(\alpha)$ is equal to the reference value $s^2$ of the cusum test of $\sigma_a = 1$ versus $\sigma_r = k^2$ (equation 3.1); and as we pointed out in chapter 2, this is the value that gives maximum discrimination when an observation has the same probability of coming from either one of the distributions. Finally when $\alpha$ tends to 1; i.e. we are certain of the presence of outliers, $s^2(\alpha)$ tends to $-\infty$, and $CS_n^+$ will tend to $\infty$.

If all the observations but one have variance $\sigma^2$, the cumulative sum chart will tend to be horizontal and close to 0 except for the outlying observation that will produce a peak in the chart. The magnitude of the jump will depend on the value of the prior probability $\alpha$ and the value of $k$.

Although no formal test of significance is given, three examples will show how the cumulative sum chart can be used as graphical procedure for the detection of outlying observations.

*Example 1:* Fifty observations were generated from a normal distribution with mean 0 and variance 1. Observation 18, however, was generated from a normal distribution with mean 0 and variance 3. Figure 3.6a shows that observation 18, although somewhat large, does not appear to be inconsistent with the remainder of the data set. Figures 3.6b and 3.6c show the cusum plot with $\alpha$ values 0.07 and 0.1 and $k = \sqrt{3}$. From both plot it is evident that observation 18, with prior probability

**Figure 3.6 a)** Fifty observations from $N(0,1)$ with observation 18 coming from a $N(0,3)$. **b-c)** CUSUM charts with $\alpha = 0.07$, $\alpha = 0.1$ and $k = \sqrt{3}$.



$\alpha$, has variance larger than 1. As expected, the cusum is flat with a peak right at observation 18. Note also how the value of $CS_n^+$ increases as $\alpha$ increases. For $\alpha = 0.05$ the cusum did not detect the outlier.

*Example 2:* Table 3.5 gives 21 observations $(x,y)$ of age at first word $(x)$ and the Gesell adaptative score $(y)$; as well as the standardized residuals from the model $y = 109.87 - 1.127x$, see Draper and John (1981).

| Case | $x$ | $y$ | Standardized Residual |
|---|---|---|---|
| 1 | 15 | 95 | 0.1890385 |
| 2 | 26 | 71 | -0.8909439 |
| 3 | 10 | 83 | -1.452367 |
| 4 | 9 | 91 | -0.8126489 |
| 5 | 15 | 102 | 0.8405769 |
| 6 | 20 | 87 | -0.03109338 |
| 7 | 18 | 93 | 0.3175747 |
| 8 | 11 | 100 | 0.2348365 |
| 9 | 8 | 104 | 0.2924541 |
| 10 | 20 | 94 | 0.6204450 |
| 11 | 7 | 113 | 1.025249 |
| 12 | 9 | 96 | -0.3472644 |
| 13 | 10 | 83 | -1.452367 |
| 14 | 11 | 84 | -1.254394 |
| 15 | 11 | 102 | 0.4209903 |
| 16 | 10 | 100 | 0.1299398 |
| 17 | 12 | 105 | 0.8051177 |
| 18 | 42 | 57 | -0.5156740 |
| 19 | 17 | 121 | 2.818831 |
| 20 | 11 | 86 | -1.068240 |
| 21 | 10 | 100 | 0.1299398 |

**Table 3.5** Age at first word $(x)$, Gesell score $(y)$ and standardized residuals.

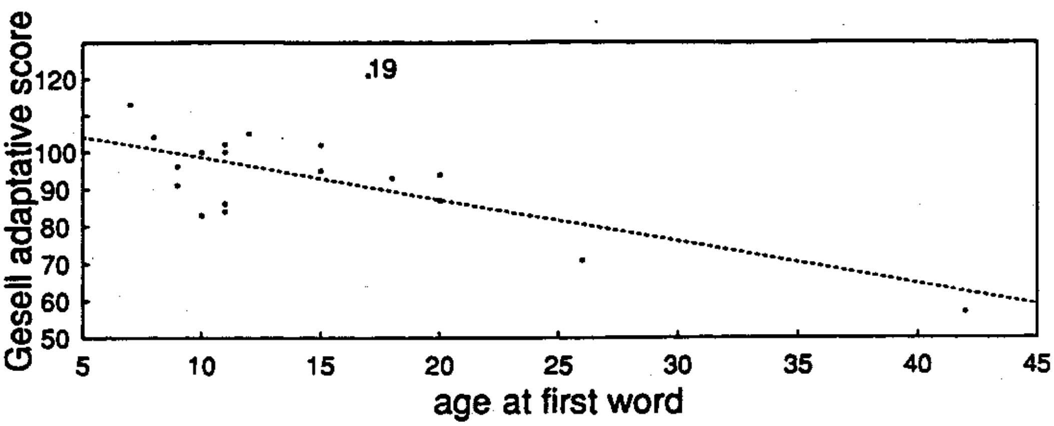
Figure 3.7a shows a plot of the data and the fitted regression line. It is clear that observation 19 stands out as being somewhat different from the rest. Several statistics for identifying outliers, see Balasooriya *et al.* (1987), reject this observation as an outlier. Draper and John also concluded, by means of the $Q_k$ statistic (John and Draper 1978), that this observation is an outlier. In this case $Q_1 = 969$, and large values of this statistic are considered to be deviant.

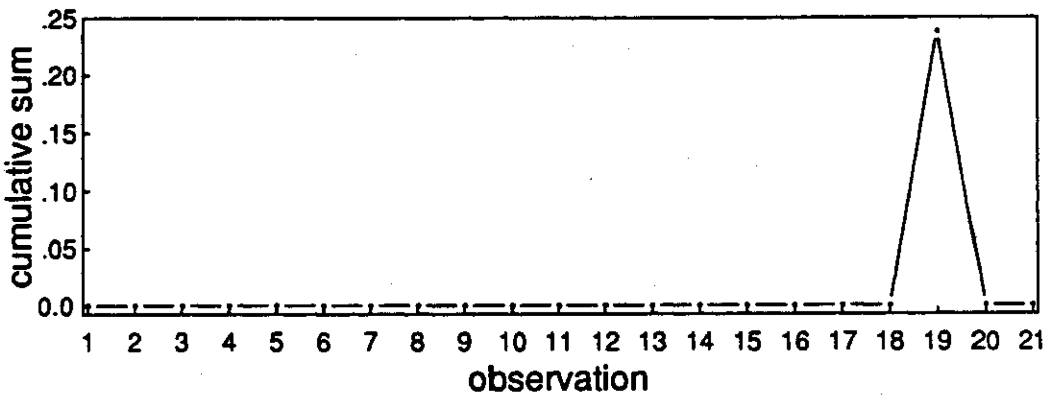
Figure 3.7b shows the cusum chart with $\alpha = 0.1$ and $k = \sqrt{2}$, and figure 3.7c shows the cusum chart with $\alpha = 0.2$ and $k = 2$. The plots show a peak at observation 19 suggesting that this data point could be an outlier.

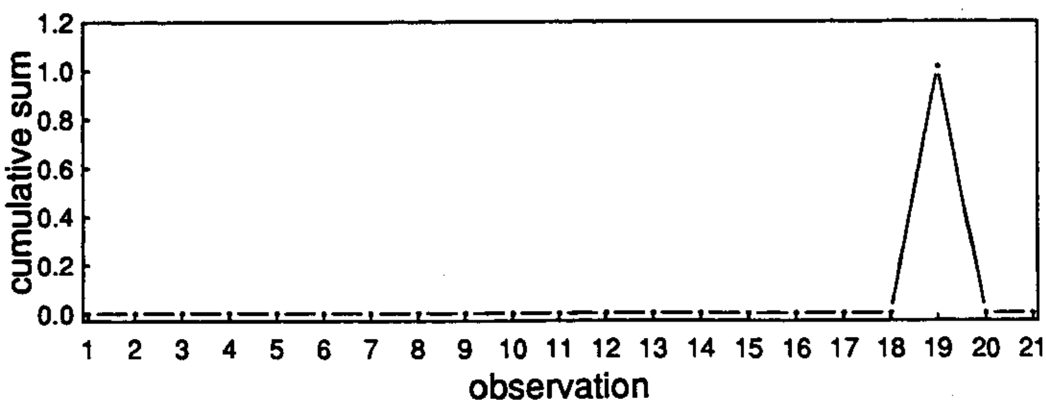
*Example 3:* The data in table 3.6 is from Snedecor and Cochran (1968), and comes from an investigation of the sources from which corn plants in Iowa soils obtain their phosphorus. The concentration in parts per million of inorganic ($x_1$) and organic ($x_2$) phosphorus in the soils and the phosphorus content ($y$) in the corn plant were measured for 18 soils.

Snedecor and Cochran after fitting a regression model of $y$ on $x_1$ and $x_2$, concluded that residual 17 was "suspiciously large" giving no explanation for its size. In a later edition of the book, Snedecor and Cochran (1980), observation 17 was not included in the data set. Lund (1975) concluded that the value of the standardized residual of 3.38 of observation 17 was significant with a p-level less that 0.01. Also the results in table 1 of Balasooriya *et al.* (1987), show observation 17 to be rejected as an outlier at 5% significance level for various statistics for identifying outliers. John and Draper's $Q_k$ statistic for $k = 1$, $Q_1 = 4313$, is extremely large

**Figure 3.7** **a)** Data from table 3.5 and regression line $y = 109.87 - 1.127x$. **b)** CUSUM chart with $\alpha = 0.1$ and $k = \sqrt{2}$. **c)** CUSUM chart with $\alpha = 0.2$ and $k = 2$



indicating again this observation as an outlier.

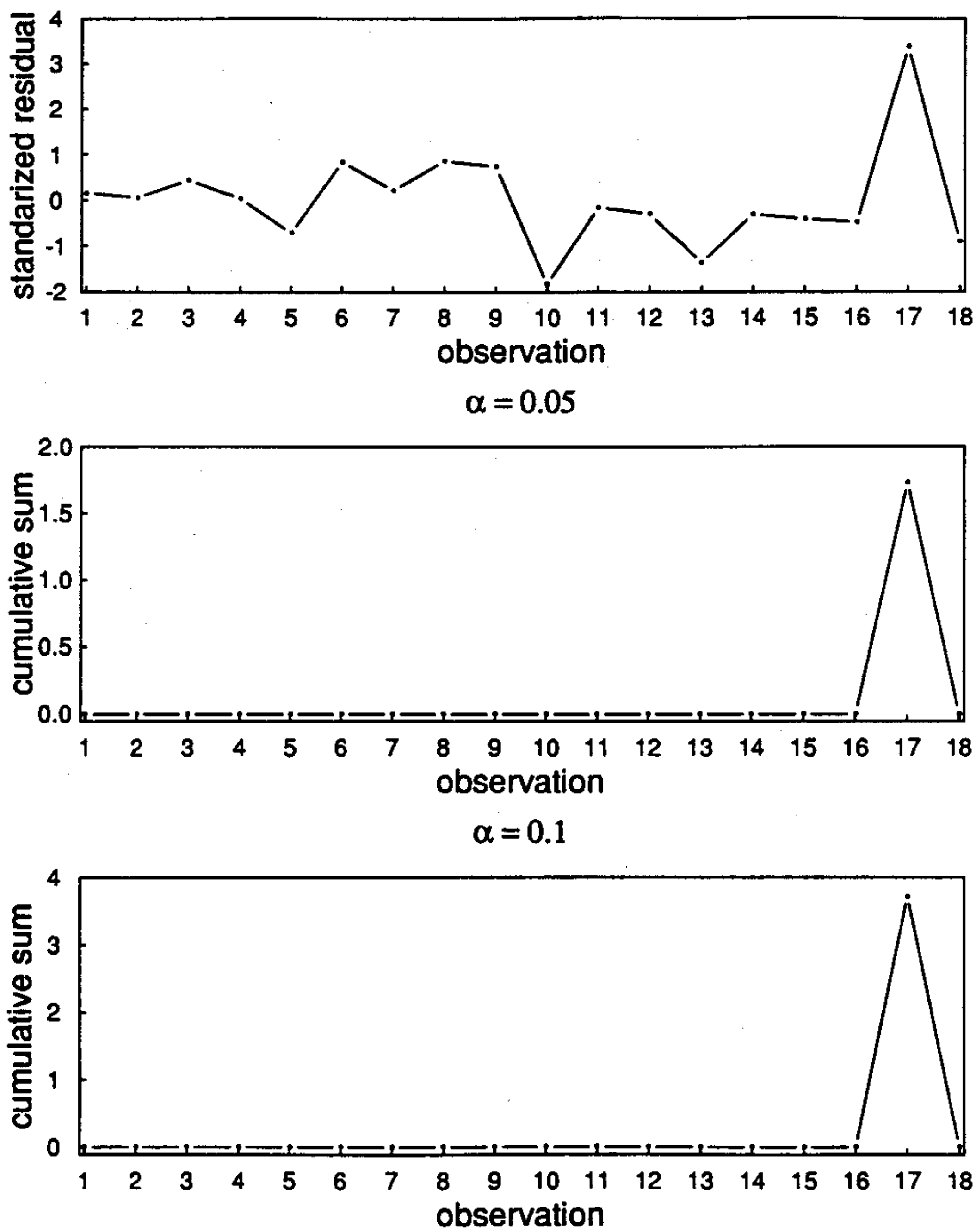
| Soil Sample | $y$ | $x_1$ | $x_2$ | Standardized residual |
|---|---|---|---|---|
| 1 | 64 | 0.4 | 53 | 0.1463247 |
| 2 | 60 | 0.4 | 23 | 0.0594694 |
| 3 | 71 | 3.1 | 19 | 0.4418922 |
| 4 | 61 | 0.6 | 34 | 0.0402264 |
| 5 | 54 | 4.7 | 24 | -0.7158810 |
| 6 | 77 | 1.7 | 65 | 0.8455418 |
| 7 | 81 | 9.4 | 44 | 0.2188530 |
| 8 | 93 | 10.1 | 31 | 0.8663325 |
| 9 | 93 | 11.6 | 29 | 0.7406044 |
| 10 | 51 | 12.6 | 58 | -1.835651 |
| 11 | 76 | 10.9 | 37 | -0.1578270 |
| 12 | 96 | 23.1 | 46 | -0.3080657 |
| 13 | 77 | 23.1 | 50 | -1.373450 |
| 14 | 93 | 21.6 | 44 | -0.3132632 |
| 15 | 95 | 23.1 | 56 | -0.4149218 |
| 16 | 54 | 1.9 | 36 | -0.4794859 |
| 17 | 168 | 26.8 | 58 | 3.380913 |
| 18 | 99 | 29.9 | 51 | -0.8975834 |

**Table 3.6** Inorganic phosphorus ($x_1$), organic phosphorus ($x_2$) and plant available phosphorus ($y$) in 18 Iowa soils.

Figure 3.8a shows the standardized residuals from the model $y = 56.25 + 1.79x_1 + 0.08x_2$ fitted to the data in table 3.6. Note that the standardized residual 17 appears to deviate markedly from the other members of the sample. Figures 3.8b and 3.8c show the cusum plots with $\alpha = 0.05$ and $\alpha = 0.1$ respectively, and $k = 2$. Even with a prior probability of 0.05, the cusum chart indicates that observation 17 is an outlier.

**Figure 3.8 a)** Standarized residuals from the model $y = 56.25 + 1.79x_1 + 0.08x_2$.

**b)** and **c)** CUSUM charts with $\alpha = 0.05$, $\alpha = 0.1$ and $k = \sqrt{2}$.



### 3.7 Non-normal Observations

In section 2.6 we discussed how the exponential power family of distributions can be used to study the effects of departure from normality, with respect to concentartion around the mean value $\mu$; i.e, peakedness, in the ARL values. The parameter $\gamma$ can be seen as a measure of kurtosis measuring the "non-normality" of the population. We also studied the $t$-distribution, that has longer tails than the normal, and we saw how CUSUM charts based on the exponential power distribution, with values of $\gamma \geq 0$, are equivalent to those obtained by using the $t$-distribution. In this section we describe how to design CUSUM charts, using the exponential power family of distributions.

Selection of the action limit $h$, as before, depends on some predetermined average run lengths at the acceptable quality level $L_a$ and the rejectable quality level $L_r$. Once the magnitude of the shift $\sigma_r/\sigma_a$ that we want to detect and the parameter $\gamma$ have been determined, five possible CUSUM charts, corresponding to the values $h = 5, 7, 10, 12, 15$, can be chosen from the tables given in the appendix. We need then an estimate of the amount of "non-normality", as measured by $\gamma$, of the population, in order to select the appropiate table.

Equation 2.31, in section 2.6.4, shows the relation between the kurtosis, $\gamma_2$, of the exponential power distribution and the parameter $\gamma$.

$$\gamma_2 = \frac{\Gamma[\frac{5}{2}(1+\gamma)]\Gamma[\frac{1}{2}(1+\gamma)]}{\Gamma[\frac{3}{2}(1+\gamma)]^2} - 3 \ .$$

Although we can not solve explicitly for $\gamma$, an estimate, $\hat{\gamma}$, of the parameter $\gamma$ as a function of the kurtosis $\gamma_2$, is given by the formula

$$\hat{\gamma} = 0.58 g_2 - 0.09 g_2^2 \ . \qquad (3.18)$$

Here $g_2$ is a sample estimate of the kurtosis; i.e.,

$$g_2 = \frac{k \sum_{i=1}^{k} (x_i - \bar{X})^4}{\left(\sum_{i=1}^{k} (x_i - \bar{X})^2\right)^2} - 3$$

For the double exponential distribution $\gamma_2 = 3$ and equation 3.18 gives $\hat{\gamma} = 0.93$, in close agreement with the actual value of $\gamma = 1$; while for the rectangular distribution $\gamma_2 = -1.2$ and $\hat{\gamma} = -0.83$.

Once the value of $\hat{\gamma}$ has been determined, the appropiate table can be used to select the parameters of the chart. Sometimes, rather than designing the whole chart, we would like to know what the performance of the chart would be if we use some parameters $h$, $\sigma_r / \sigma_a$ and $\gamma$. In this case, an approximation formula for the ARL can give us preliminary information about the performance of the control scheme.

Following the ideas described in section 3.5, an approximation formula of the iterated logarithm of ARL, *loglog* $(L)$, as a function of the action limit $h$, the shift in process variance $\sigma_r/\sigma_a$ and the "non-normality" parameter $\gamma$, was obtained by fitting the model

$$loglog(L_i) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \beta_4 x_1^2 + \beta_5 x_2^2 \\ + \beta_6 x_3 + \beta_7 x_3^2 + \beta_8 x_1 x_3 + \beta_9 x_2 x_3 + \varepsilon . \quad (3.19)$$

Where $x_1 = log(h)$, $x_2 = (\sigma_r/\sigma_a)^{-1}$, $x_3 = \gamma$ and $L_i$, $i = a,r$, denotes the average run length at the acceptable and rejectable quality levels.

The table below shows the estimates and standard errors for the parameter of the model given by equation (3.19) for $L_a$ and $L_r$.

| | $L_a$ | | $L_r$ | |
|---|---|---|---|---|
| Parameter | Estimate | Std. Err. | Estimate | Std. Err. |
| $\beta_0$ | 1.125 | 0.042 | -1.560 | 0.032 |
| $\beta_1$ | 0.325 | 0.033 | 0.363 | 0.023 |
| $\beta_2$ | -0.205 | 0.060 | 3.550 | 0.052 |
| $\beta_3$ | -0.268 | 0.016 | -0.040 | 0.012 |
| $\beta_4$ | 0.071 | 0.007 | -0.011 | 0.005 |
| $\beta_5$ | -0.029 | 0.042 | -1.285 | 0.037 |
| $\beta_6$ | 0.199 | 0.012 | 1.200 | 0.010 |
| $\beta_7$ | -0.331 | 0.003 | -0.494 | 0.003 |
| $\beta_8$ | 0.299 | 0.005 | 0.104 | 0.004 |
| $\beta_9$ | -0.277 | 0.011 | -0.900 | 0.010 |

**Table 3.7** Parameter estimates for the model in equation 3.19.

As before the approximation is very good especially for the $L_r$ values. The $L_a$ approximation tends to underestimate the ARL values.

### 3.8 Summary

In this chapter we have seen how the contour nomogram can be used to design CUSUM charts for monitoring increases or decreases in process variance. Once the average run lengths at the acceptable quality level $\sigma_a$, denoted by $L_a$, and at the rejectable quality level $\sigma_r$, denoted by $L_r$, have been selected, we use the nomogram to select the action limit $h$ that yields the desired $L_a$ and $L_r$.

For monitoring shifts in process variance of magnitude $\sigma_r/\sigma_a$, a horizontal line can be drawn in the contour nomogram at the point $\sigma_r/\sigma_a$, to obtain several values of the decision interval $h$, giving various possible combinations of $L_a$ and $L_r$.

It is assumed that the process mean is known. The effect of using the sample average $\bar{X}$ is that of increasing the average run length.

A CUSUM chart for monitoring shifts in both directions can be constructed by plotting in the same chart the two cumulative sums $CS_k^+$ and $CS_k^-$, along with the decision intervals $h_i$ and $h_d$ obtained from the contour nomogram. The chart is similar to the Shewhart chart with two horizontal action limits; although for most applications the limits will not be symmetric. An out of control signal will be given whenever $CS_k^+ \geq h_i$ or $CS_k^- \leq h_d$.

In the following pages an outline is given that summarizes the procedure for designing CUSUM charts for monitoring variances.

## An Outline For Implementing CUSUM Charts For Monitoring Variance

1. Let $x_k$ be the outcomes of a process whose variability we want to monitor.

2. Determine the Quality Levels $\sigma_a$, $\sigma_r$ and $\sigma_d$.

   a) $\sigma_a$ is the level at which the process is currently running that you wish to monitor.

   b) $\sigma_r > \sigma_a$ is the smallest increase that is important enough to be detected quickly.
   This defines the out-of-control region $M_r = \{ \sigma \ : \ \sigma \geq \sigma_r \}$

   c) $0 < \sigma_d < \sigma_a$ is the largest decrease that is important enough to be detected quickly.
   This defines the desirable region $M_d = \{ \sigma \ : \ 0 < \sigma \leq \sigma_d \}$

3. Calculate the Reference Values $s_i^2$ and $s_d^2$ using the formulas

$$s_i^2 = \frac{\log\left(\frac{\sigma_r^2}{\sigma_a^2}\right)}{\frac{1}{\sigma_a^2} - \frac{1}{\sigma_r^2}} \qquad \text{and} \qquad s_d^2 = \frac{\log\left(\frac{\sigma_d^2}{\sigma_a^2}\right)}{\frac{1}{\sigma_a^2} - \frac{1}{\sigma_d^2}}$$

4. Determine the Action Limits $h_i$ and $h_d$ that Yield the Desired Average Run Lengths $L_a$ and $L_r$.

a) Draw a horizontal line in the Contour Nomogram at the point $\frac{\sigma_r}{\sigma_a}$.

This gives various possible combinations of $\frac{h}{\sigma_a^2}$, $L_a$ and $L_r$.

b) The action limit for monitoring increases in variance is $h_i = h$.

c) Draw a horizontal line in the Contour Nomogram at the point $\frac{\sigma_a}{\sigma_d}$.

This gives various possible combinations of $\frac{h}{\sigma_a^2}$, $L_a$ and $L_r$.

d) The action limit for monitoring decreases in variance is $h_d = -\frac{\sigma_d^2}{\sigma_a^2}h$.

5. Compute the CUSUMS $CS_k^+$ and $CS_k^-$.

For each observation $x_k$, starting with $CS_0^+ = CS_0^- = 0$, compute:

a) Increases in variance

$$r_k^+ = [(x_k - \mu)^2 - s_i^2],$$

$$CS_k^+ = max(0\,,\, CS_{k-1}^+ + r_k^+)$$

b) Decreases in variance

$$r_k^- = [(x_k - \mu)^2 - s_d^2],$$

$$CS_k^- = min(0\,,\, CS_{k-1}^- + r_k^-)$$

6. Tabulate or plot sequentially with $k$ the CUSUM $CS_k^+$ and $CS_k^-$.

a) A control chart is set up with action limits placed at $h_i$ and $h_d$ to monitor upward shifts with $CS_k^+$ and downward shifts with $CS_k^-$ respectively. Note that, as opposed to charts for the mean, the limits are not symmetric.

b) Watch the progress of the $CS_k^+$ and $CS_k^-$ values in the chart.

c) An increase in process variance is signalled whenever

$$CS_k^+ \geq h_i$$

d) A decrease in process variance is signalled whenever

$$CS_k^- \leq h_d$$

7. If the mean is not known replace $\mu$ by $\overline{X}$ in step 5.

a) Multiply the $L_a$ obtained from the nomogram by 1.82

b) Multiply the $L_r$ obtained from the nomogram by 1.22

8. For samples of size $m \neq 1$ use

a) Reference values: $\dfrac{s_i^2}{m}$ and $\dfrac{s_d^2}{m}$

b) Action limits: $\dfrac{h_i}{m}$ and $\dfrac{h_d}{m}$

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