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Report No. 89

Charts for Optimal Feedback Control with Recursive Sampling and Adjustment

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Charts for Optimal Feedback Control with Recursive Sampling and Adjustment

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ABSTRACT

A cost model proposed by Box and Jenkins (1963) and later generalized by Box and Kramer (1992) for obtaining minimum cost for feedback control of processes is considered. Unfortunately, it is sometimes difficult to assign values to the costs of making an adjustment, of taking a sample and of being off target as is required by their approach. An alternative that avoids the direct assignment of values to these costs is discussed in this paper, and charts are provided to aid in choosing a reasonable scheme. For different values of the action limit and the non-stationarity measure, it is possible to compute an envelope of optimal schemes from which a choice may be made by judging the disadvantage of an increased mean square deviation against the advantage of having to take samples less frequently and/or increasing the average adjustment interval.

KEYWORDS: *Action limit; Automatic process control; Average adjustment interval; Cost; Mean square deviation; Monitoring interval.*

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A cost model proposed by Box and Jenkins (1963) and later generalized by Box and Kramer (1992) for obtaining minimum cost for feedback control of processes is considered. Unfortunately, it is sometimes difficult to assign values to the costs of making an adjustment, of taking a sample and of being off target as is required by their approach. An alternative that avoids the direct assignment of values to these costs is discussed in this paper, and charts are provided to aid in choosing a reasonable scheme. For different values of the action limit and the non-stationarity measure, it is possible to compute an envelope of optimal schemes from which a choice may be made by judging the disadvantage of an increased mean square deviation against the advantage of having to take samples less frequently and/or increasing the average adjustment interval.

1. INTRODUCTION

Box and Jenkins (1963) considered a model for feedback control which assumed that the cost of being off-target was a quadratic function of the deviation of the observation z_t from the target value T, and that when an adjustment was finally made, this would result in an additional fixed cost. It has recently been argued (Box and Kramer 1992) both on theoretical grounds and on the basis of actual experience that the IMA(0,1,1) time series model defined by $z_t - z_{t-1} = a_t - \theta a_{t-1}$ is frequently valuable to approximate the process disturbance (see also Box and Jenkins 1970). These authors showed, on the assumptions mentioned, that the minimum cost scheme would be of the form where the EWMA of past observations

$$\hat{z}_t = \lambda\left(z_{t-1} + \theta z_{t-2} + \theta^2 z_{t-3} + \ldots\right), \text{ with } 0 \le \lambda \le 1,$$

would be plotted between two parallel lines, and an action $T - \hat{z}_{t+1}$ would be taken as soon as $\hat{z}_{t+1}$ crosses the lines $|\hat{z}_{t+1} - T| = L$. For any proposed scheme the average interval between adjustments that we call AAI may be computed.

A different approach was adopted by other writers including Taguchi (1981) and Srivastava and Wu (1991) who plotted the actual value z_t instead of the EWMA $\hat{z}_{t+1}$. On the assumptions mentioned above (except in the case where the non-stationarity measure $\lambda = 1-\theta$ is equal to one, when the two forms are identical) their schemes are not optimal. Luceño (1992) showed that the main reason for this is because the optimum adjustment is in fact $T - \hat{z}_{t+1}$ rather than $T - z_t$, and that the cost of regulation applying the policy based on z_t can be considerably larger than that based on $\hat{z}_{t+1}$.

Recently Box and Kramer (1992) generalized the model by considering also the cost of *monitoring* the process, i.e. the cost of taking each observation. In particular, supposing initially that the process was sampled at unit intervals, they considered the possibility of sampling at intervals m units apart, where m is an integer.

These authors arrive at the following overall cost function:

$$C = \frac{C_A}{m\, h\left[L/(\lambda_m \sigma_m)\right]} + \frac{C_M}{m} + C_T\left\{\frac{\theta}{\theta_m} + m\lambda^2 g\left[L/(\lambda_m \sigma_m)\right] - \frac{(m-1)\lambda^2}{2}\right\},$$

where $h(B)$ and $g(B)$ are functions that have been approximated by Box and Jenkins (1963), Lucas and Croisier (1982), Crowder (1987), Kramer (1989) and Srivastava and Wu (1991), C_T is the cost of being off target for one time interval by an amount of $\sigma_a^2 = \text{Var}(a_t)$, C_A and C_M are the fixed costs incurred each time the process is adjusted or observed, respectively, and $\lambda_m = 1-\theta_m$ and σ_m are given by

$$\lambda_m^2 \sigma_m^2 = m\lambda^2\sigma_a^2 \text{ and } \theta_m \sigma_m^2 = \theta\sigma_a^2.$$

While λ and σ_a may be estimated using past plot data (see Box and Jenkins 1970), a difficulty about this procedure has been that one has to know values for the cost C_A of making an adjustment, C_M of taking a sample, and the constant C_T that determines how rapidly the quadratic loss function increases.

In the simpler case where the monitoring interval was regarded as fixed, it was pointed out recently by Box (1992) how the need to estimate individual cost could be avoided. A base case was considered where L=0 and samples and adjustments where made at unit intervals. This is appropriate for the situation where there are no costs other than that of being off target. A table was presented which, for given nonzero values of L, shows the increasing AAI together with the corresponding increasing mean square error compared with the base scheme. An empirical judgement could then be made of a reasonable compromise.

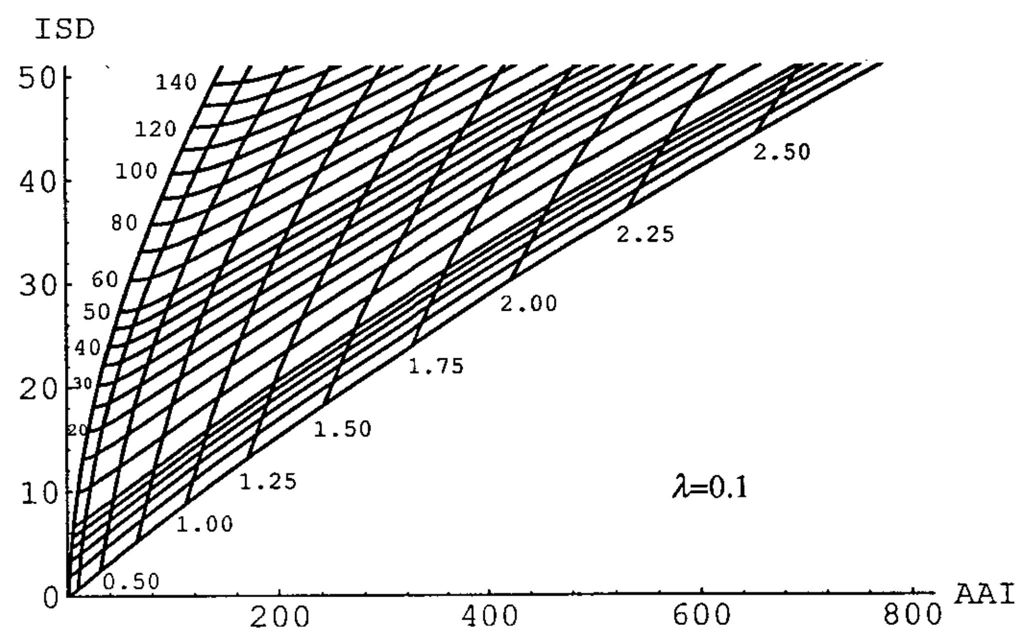
The main purpose of this paper is to reuse the results of Box and Kramer (1992) to get charts of optimal schemes from which one could also choose m in a similar empirical way. That is, on the basis of how much the mean square error would need to increase to achieve the advantage of *taking samples* and making adjustments less frequently. The charts required to present the optimal schemes are described in Section 2 and some examples are given in Section 3.

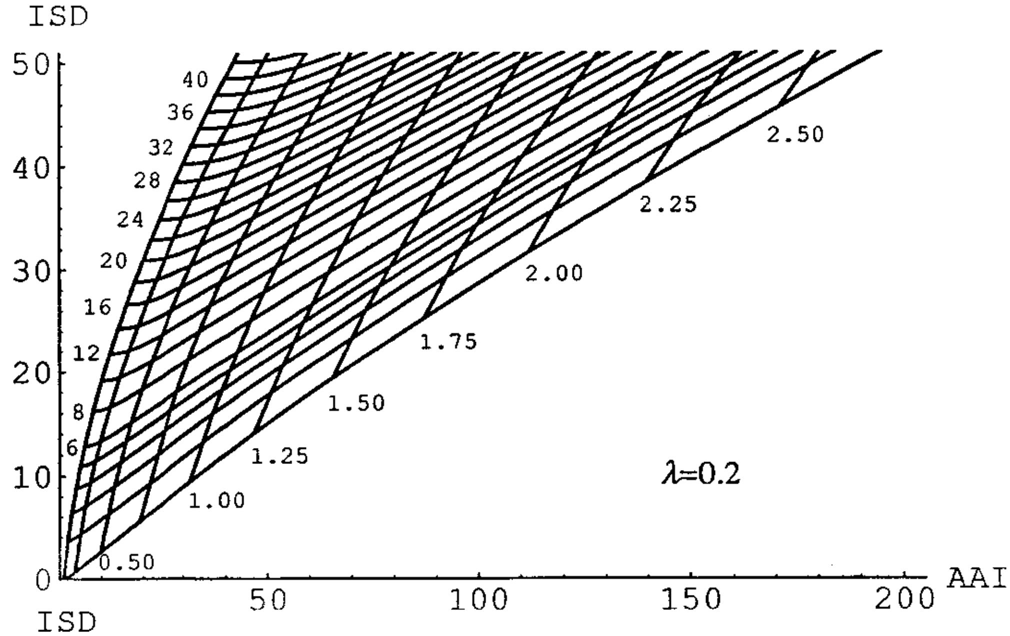
2. DESCRIPTION OF CHARTS

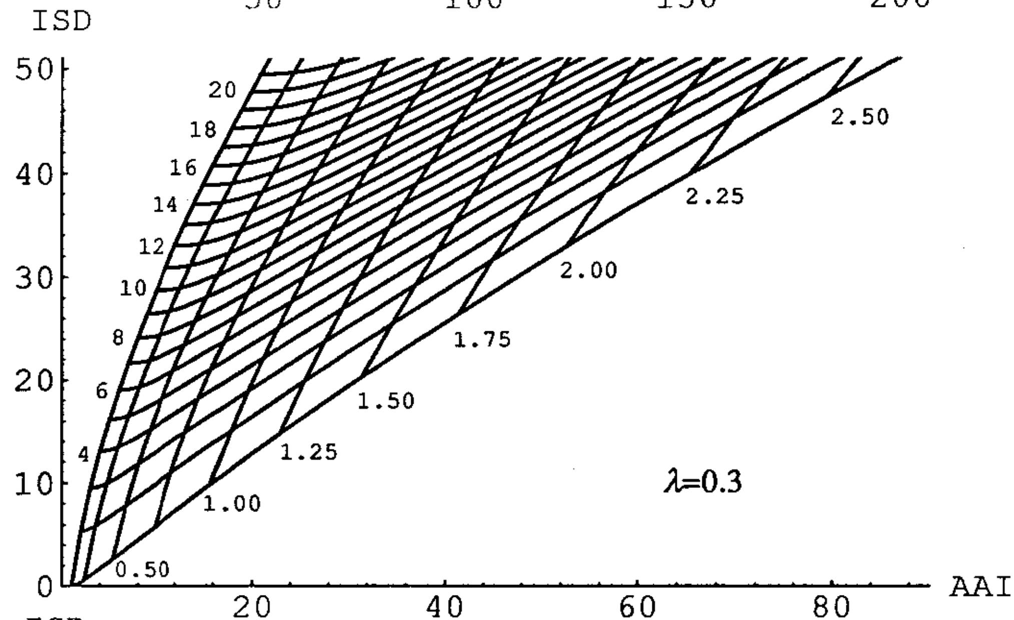
Charts 1-6 of Figure 1 give the values of the average adjustment interval (AAI) in terms of the original units of time and the percent increase in standard deviation (ISD) with respect to σ_a corresponding to values of the non-stationarity measure λ=0.1 (0.1) 0.6, the standardized action limit L/σ_a=0.0 (0.25) 2.5, and the monitoring interval m=1, ..., 140. The charts are limited to cover only small to moderate values of the percent increase in standard deviation so that the greater values of m only appear combined with the smaller values of λ. Thus m=1 (1) 6, 10 (5) 50, 60 (10) 140 for λ=0.1; m=1 (1) 5, 6 (2) 42 for λ=0.2; m=1 (1) 21 for λ=0.3; m=1 (1) 13 for λ=0.4; m=1 (1) 9 for λ=0.5; and m=1 (1) 7 for λ=0.6.

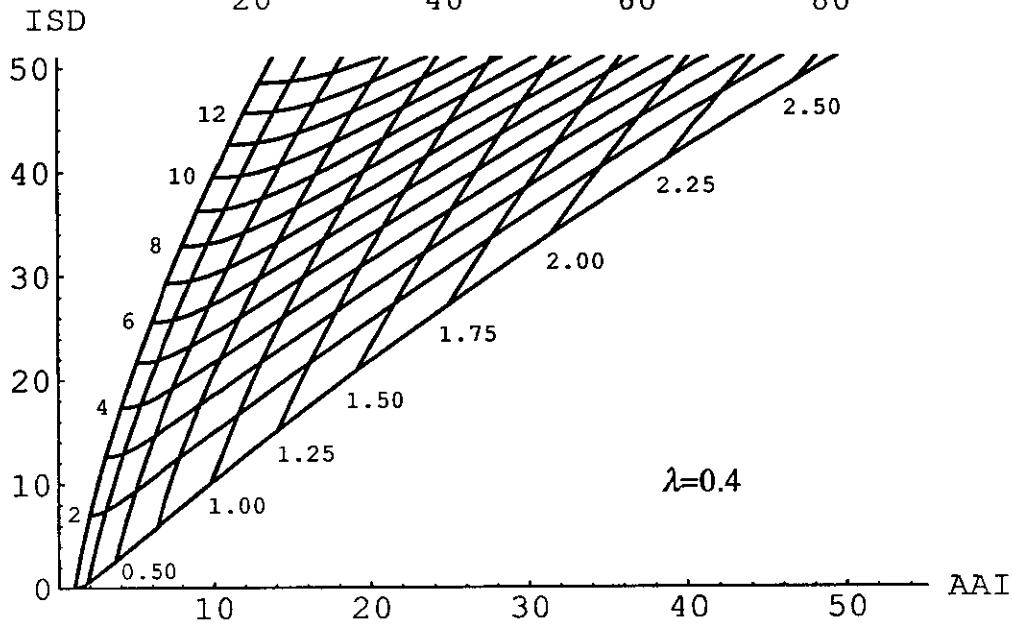
3. EXAMPLE OF THE USE OF THE CHARTS

Following Box and Kramer (1992), suppose we need to control a quality characteristic subject to a disturbance which was identified and estimated as an IMA(0,1,1) with λ=0.3 and σ_a=3.0. Suppose that the target value is 340 and the specification limits are 325 and 355, i.e. 340±15.




ISD
50
40
30
20
10
0
40
36
32
28
24
20
16
12
8
6
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.50
λ=0.2
AAI
50
100
150
200


ISD
50
40
30
20
10
0
20
18
16
14
12
10
8
6
4
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.50
λ=0.3
AAI
20
40
60
80


ISD
50
40
30
20
10
0
12
10
8
6
4
2
2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.50
λ=0.4
AAI
10
20
30
40
50

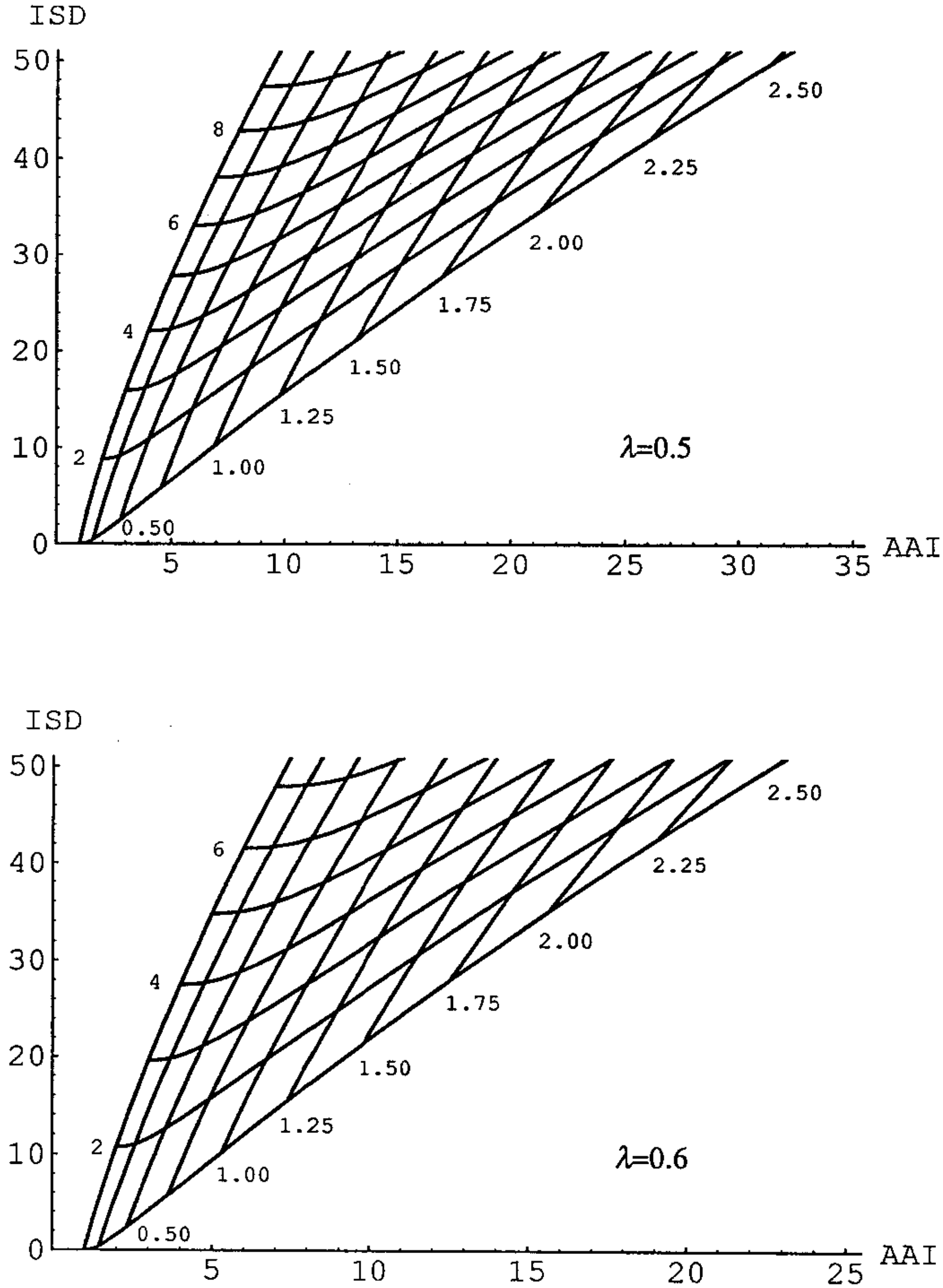


Figure 1. *Average adjustment interval and percent increase in standard deviation with respect to σ_a for λ=0.1 (0.1) 0.6, L/σ_a=0.0 (0.25) 2.5 and several values of m from 1 to 140. The AAI is in terms of the original time units and not in terms of the time units of the sampled process.*

If the action limit is L=3.0 and the monitoring interval is m=10, the average adjustment interval and the percent increase in standard deviation are equal to 29.5 and 36.5% respectively. This is represented by point A in Figure 2.

Then we might increase the AAI, for example, to 40.3 with nearly the same ISD, 37.8, by taking $L=1.4 \times 3=4.2$ and observing the process more frequently, m=7 (point B in Figure 2).

We also could reduce the ISD to 24.8, keeping almost the same AAI, 29.4, by taking $L=1.3 \times 3=3.9$ and m=3 (point C in Figure 2). Or, by using $L=0.55 \times 3=1.65$, we could increase the monitoring interval to m=20 with a similar AAI provided that we are ready to assume an ISD of 49.1 (point D in Figure 2).

In addition, if the value of λ is only known to be in the interval 0.30±0.10, and L=3.0 and m=10, the AAI and ISD will be in the intervals (22.1, 49.3) and (27.8, 46.1) respectively.

In this way, a number of alternative schemes can be brought out in an envelope and we can then ask the people involved in the process to make a choice. The charts provide a way of seeing what the increase in mean square error and the average adjustment interval are as a function of how frequently we need a sample and where the action limits are.

4. THE COST FUNCTION UNDER ALTERNATIVE MODELS

For the charts to be useful in practice, one needs to know how sensitive are the results to the assumptions made. In this section we give this question some consideration. Whether the IMA(0,1,1) time series model or a different model is used, the overall cost function can be computed if the AAI and ISD, or equivalently the Mean Square Deviation (MSD), can be calculated. The cost is then given by

$$C = \frac{C_M}{m} + \frac{C_A}{m(AAI)} + \frac{C_T}{\sigma_a^2}(MSD),$$

where

$$MSD = \frac{1}{m(AAI)} E\left[\sum_{j=0}^{n-1}\sum_{k=1}^{m}\left(z_{mj+k} - T\right)^2\right].$$

Moreover, the AAI and the numerator of the MSD can be evaluated using an approach similar to that used by Crowder (1987) in the context of Statistical Process Control. This evaluation can be done provided that the action criteria be $|\hat{z}_t - T| \geq L$, the *computed* one step ahead forecast be updated with $\hat{z}_t = \hat{z}_{t-1} + \lambda a_{t-1}$, and the pdf of a_{t-1} be known.

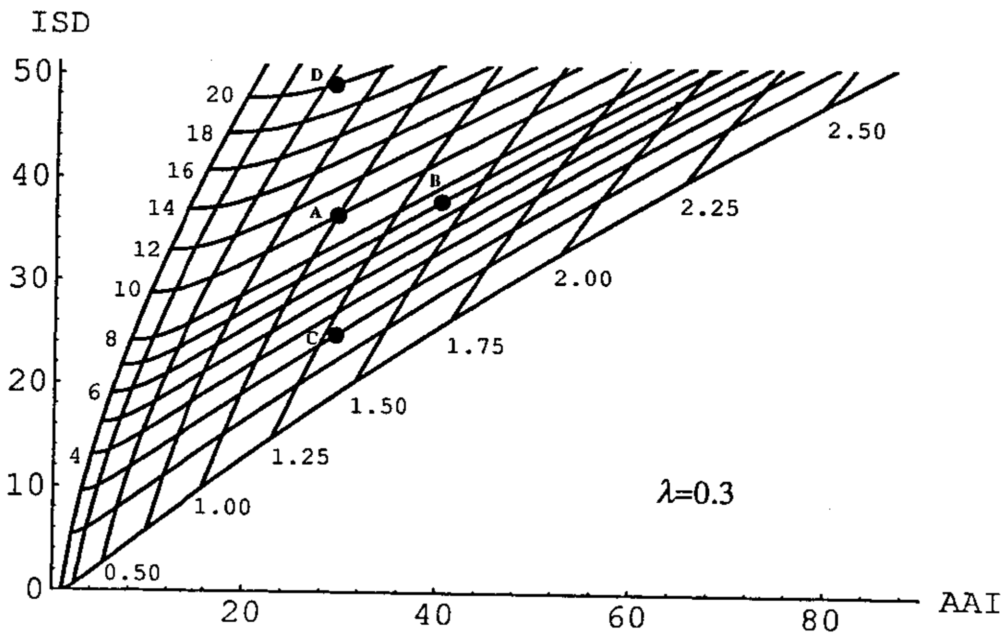


Figure 2. *Average adjustment interval and percent increase in standard deviation with respect to* σ_a *for* $\lambda = 0.3, L/\sigma_a = 0.0\,(0.25)\,2.5$ *and several values of m from 1 to 20. Points A-D correspond to the example of Section 3. The AAI and ISD for these points are* (29.5, 36.5), (40.3, 37.8), (29.4, 24.8), *and (29.1, 49.1) respectively.*

Let $A(u,L)$ be the average remaining adjustment interval after t–2, given that the computed one step ahead forecast is $T+u$ and the action limit is L. Then, at time $t-1$, we will conclude that the process is out of control and call for a control action if $|\hat{z}_t - T| \geq L$. In this case, the remaining adjustment interval after t–2 will be equal to 1. If $|\hat{z}_t - T| < L$, the process will continue in control for at least one more unit of time, the computed one step ahead forecast will change from $\hat{z}_{t-1}$ to $\hat{z}_t = \hat{z}_{t-1} + \lambda a_{t-1}$, where a_{t-1} is the next *computed* shock, and the remaining adjustment interval after t-2 will be $1 + A(\hat{z}_t, L)$. Therefore,

$$\begin{aligned} A(u,L) &= 1 + \int_{|u+\lambda a|<L} A(u+\lambda a, L) f(a)\,da \\ &= 1 + \frac{1}{\lambda}\int_{-L}^{L} A(x,L) f\left(\frac{x-u}{\lambda}\right) dx, \end{aligned} \tag{4.1}$$

where $f(.)$ is the pdf of the computed shock a_{t-1}.

Analogously, the numerator of the MSD satisfies the integral equation

$$M(u,L) = u^2 + \sigma_a^2 + \frac{1}{\lambda}\int_{-L}^{L} M(x,L) f\left(\frac{x-u}{\lambda}\right) dx. \tag{4.2}$$

Equations (4.1) and (4.2) are Fredholm integral equations and can be solved numerically. A detailed discussion of methods used to obtain numerical solutions to integral equations is given by Baker (1977). When the IMA(0,1,1) model is true, $f(.)$ is the normal density function with mean zero and variance σ_a^2. Otherwise $f(.)$ is the density function of the next computed shock which depends on the true disturbance model.

A simple example of the use of (4.1) and (4.2) can be shown assuming that $f(.)$ is the uniform distribution in the range $\left[-(3\sigma_a^2)^{1/2}, (3\sigma_a^2)^{1/2}\right]$. In this case, the solution of the integral equations is particularly simple if $L/(\lambda\sigma_a) \leq 3^{1/2}/2$, and the AAI and MSD are then given respectively by

$$AAI = \left[1 - \left(\frac{L}{\lambda\sigma_a}\right)\frac{1}{\sqrt{3}}\right]^{-1},$$

$$MSD = \sigma_a^2 + \lambda^2\sigma_a^2\left(\frac{L}{\lambda\sigma_a}\right)^3 \frac{1}{3\sqrt{3}}.$$

Some values of the AAI and ISD under the IMA(0,1,1), which have been computed solving (4.1) and (4.2) numerically for λ=1 and σ_a=1, and the uniform models are given in Table 1.

Table 1

Average adjustment interval and percent increse in standard deviaiton with respect to σ_a for λ=1 and σ_a=1, under the IMA (0, 1, 1) and uniform models.

$L/(\lambda\sigma_a)$	*IMA MODEL*		*UNIFORM MODEL*	
	AAI	*ISD*	*AAI*	*ISD*
0.00	1.0000	0.0000	1.0000	0.0000
0.25	1.2454	0.2039	1.1687	0.1502
0.50	1.6070	1.5167	1.4058	1.1957
0.75	2.1150	4.5233	1.7637	3.9803

Barnard (1959) proposed a model for process control in which there is a Poisson process of "jumps" occurring at a rate δ jumps per unit time, such that at each jump the process mean is shifted by an amount b which is normally distributed with mean zero and variance σ_b^2. In addition, there is a white noise process that is added to the mean and has variance σ_a^2.

In Barnard's model, the disturbance is assumed to be generated by the process $z_t = \mu_t + a_t$, where a_t has a $N(0,\sigma_a^2)$ distribution, $\mu_t = \mu_{t-1} + \varepsilon_t$, and ε_t is the sum of k independent $N(0,\sigma_b^2)$ random variables, k being an independent Poisson random variable with mean δ for each time t. Clearly the variance of ε_t is $\sigma_\varepsilon^2 = \delta\sigma_b^2$

If a process generated using the Barnard model is controlled with the action criteria $|\hat{z}_t - T| \geq L$ where the computed one step ahead forecast is updated with the formula $\hat{z}_t = \hat{z}_{t-1} + \lambda a_{t-1}$, with λ=1–θ, the variance of the next computed shock is given by

$$Var(z_t - \hat{z}_t) = \frac{2\sigma_a^2}{1+\theta} + \frac{\delta\sigma_b^2}{1-\theta}. \tag{4.3}$$

It can be easily shown that if $\delta\sigma_b^2/\sigma_a^2 = \lambda^2/\theta$, the autocorrelation structures of the IMA(0,1,1) with smoothing constant θ and the Barnard processes are the same. The variance of the noise in this IMA model is $\sigma^2 = \sigma_a^2/\theta$ which coincides with the variance of the next computed shock given by (4.3) when $\delta\sigma_b^2/\sigma_a^2 = \lambda^2/\theta$.

The distribution of next computed shock for the Barnard's model is a mixture of infinite normal distributions with mean zero and increasing variances. For this distribution, the equations (4.1) and (4.2) become very cumbersome to solve.

An easier model can be found assuming that the distribution of the next computed shock is a mixture of the $N(0, s\sigma_a^2)$ and the $N(0, s\sigma_a^2+\sigma_b^2)$ with probabilities $1-\alpha$ and α respectively. The variance of the next computed shock is then $s\sigma_a^2+\alpha\sigma_b^2$. For δ small, if $\alpha=\delta/(1-\theta^2)$ and $s=2/(1+\theta)$, the variance of the next computed shock is the same for the Barnard and for the normal-mixture models.

Table 2 gives the AAI and MSD obtained by simulation with the Barnard's model, and the AAI and MSD obtained solving equations (4.1) and (4.2) numerically using the normal-mixture and the IMA(0,1,1) models with corresponding values of λ, σ, α, and s for $\sigma_a^2=1$ and several values of $L/(\lambda\sigma)$, δ and σ_b^2. The results for the Barnard's and normal-mixture models are quite close to those found using the IMA(0,1,1) time series model.

Table 2

Average adjustment interval and mean square deviation under the Barnard, normal-mixture, and IMA(0,1,1) models with the same variance of the next computed shock, for $\sigma_a^2=1$ and several values of $L/(\lambda\sigma)$, δ, and σ_b^2. The corresponding values of λ, σ, α, and s satisfy $\delta\sigma_b^2/\sigma_a^2=\lambda^2/\theta$, $\sigma_a^2/\theta=\sigma^2$, $\alpha=\delta/(1-\theta^2)$, and $s=2/(1+\theta)$. By definition $g(.)=(MSD/\sigma^2-1)/\lambda^2$ so that $1+g(.)$ is equal to the MSD when $\lambda=1$ and $\sigma=1$.

$L/(\lambda\sigma)$	δ^{-1}	σ_b^2	*BARNARD*		*NORMAL-MIXTURE*		*IMA*	
		$(\sigma_a^2=1)$	*Simulation*		*Numerical*		*Numerical*	
			AAI	$1+g(.)$	AAI	$1+g(.)$	AAI	$1+g(.)$
4.00	64.00	4.00	29.417	3.039	21.416	4.226	21.023	4.266
		1.00	24.111	3.221	21.099	4.241		
		0.25	21.806	3.871	21.033	4.244		
	16.00	4.00	28.143	3.384	21.654	4.222		
		1.00	23.864	3.833	21.114	4.239		
		0.25	21.964	3.967	21.042	4.243		
	4.00	4.00	24.354	3.956	21.671	4.221		
		1.00	22.440	4.101	21.148	4.236		
		0.25	21.957	4.022	21.035	4.243		
2.00	64.00	4.00	7.792	1.481	7.116	1.861	6.899	1.863
		1.00	7.172	1.262	6.915	1.866		
		0.25	6.918	2.093	6.877	1.868		
	16.00	4.00	8.462	1.451	7.279	1.852		
		1.00	7.224	1.790	6.943	1.864		
		0.25	7.004	1.814	6.880	1.868		
	4.00	4.00	8.442	1.634	7.320	1.843		
		1.00	7.419	1.710	6.955	1.862		
		0.25	7.011	1.916	6.879	1.868		
1.00	64.00	4.00	2.901	1.022	2.893	1.195	2.782	1.191
		1.00	2.824	1.049	2.804	1.193		
		0.25	2.793	1.365	2.782	1.193		
	16.00	4.00	3.043	1.122	2.994	1.196		
		1.00	2.835	1.111	2.824	1.193		
		0.25	2.791	1.290	2.785	1.193		
	4.00	4.00	3.220	1.135	3.088	1.195		
		1.00	2.882	1.134	2.839	1.193		
		0.25	2.797	1.102	2.784	1.193		
0.50	64.00	4.00	1.637	1.059	1.645	1.032	1.607	1.031
		1.00	1.615	1.037	1.616	1.031		

	0.25	1.613	0.956	1.608	1.031
16.00	4.00	1.663	1.046	1.682	1.033
	1.00	1.620	1.087	1.624	1.031
	0.25	1.606	1.063	1.609	1.031
4.00	4.00	1.741	1.010	1.732	1.034
	1.00	1.636	1.012	1.632	1.032
	0.25	1.603	1.126	1.609	1.031

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