An Approach to the Jacobian Conjecture in Two Variables

Ryan Bruner, Mathematics
Steven J. Rosenberg, Ph.D. Department of Mathematics and Computer Science

ABSTRACT

We will make strides toward proving the Jacobian Conjecture in two variables when the degrees of the polynomials defining the endomorphism are relatively prime. We will use techniques that are less elementary than are strictly necessary to prove this case in hope of generalizing the proof to the case when these degrees are not relatively prime.

1 Introduction

The Jacobian Conjecture is a major outstanding conjecture in algebra from 1939 when it was first stated by mathematician Ott-Heinrich Keller. The Jacobian Conjecture says if you form a ring endomorphism σ on a polynomial ring in n variables over a field with characteristic zero, then the ring endomorphism is bijective if and only if the determinant of the jacobian of σ is a nonzero element of the given field. To date despite multiple published attempted proofs the conjecture remains unproven.

2 Notation and Definitions

Commutative Ring with 1 [Ei]

A *commutative ring with 1* is an additive abelian group R with an operation $(a,b) \mapsto ab$ referred to as multiplication such that $\forall a,b,c \in R$:

$$(ab)c = a(bc)$$

$$ab = ba$$

$$a(b+c) = ab + ac$$

$$1a = a1 = a$$

In our work we will use ring to mean a commutative ring with 1.

Polynomial Ring [La]

A polynomial ring $k[x]$ is a set of all possible polynomials with coefficients from the integral domain k and the place holder or variable x.

Given the polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$

we define the sum to be

$$f(x) + g(x) = \sum_{k=0}^{\max(m,n)} (a_k + b_k) x^k$$

we define the product to be

$$f(x)g(x) = \sum_{k=0}^{m+n} \left(\sum_{i+j=k} a_i b_j \right) x^k$$

A polynomial ring in n variables we can write as $k[x_1, ..., x_n] := (k[x_1, ..., x_{n-1}])[x_n]$.

Ring Homomorphism [La]

Let A, B be rings, and let $f : A \to B$ be a function. We say that f is a *ring homomorphism* if $f(1_A) = 1_B$ and $\forall a, b \in A$,

$$f(a+b) = f(a) + f(b)$$

and

$$f(ab) = f(a)f(b).$$

Jacobian [Ab]

Let F, G be polynomials in variables x and y, then $J(F,G)$, the jacobian of F, G relative to x, y, is defined by

$$J(F,G) = F_x G_y - G_x F_y$$

The Jacobian Conjecture in two variables

Let k be a field with characteristic zero. Let $\phi : k[x,y] \to k[x,y]$ be a ring endomorphism, fixing k pointwise, and $f = \phi(x)$, $g = \phi(y)$. Then ϕ is bijective if and only if $|J(f,g)| \in k^{\times}$.

Algebraic vs. Transcendental

Suppose R and S are rings and that S is a subring of R. Then we say that R extends S. Let $\alpha \in R$. Then:

Definition: α is called *algebraic* over S if $\exists f \in S[x] - \{0\}$ s.t. $f(\alpha) = 0$.

Definition: α is called *integral* over S if $\exists f \in S[x] - \{0\}$ s.t. $\mathrm{f}(\alpha) = 0$ and f is monic.

Definition: α is said to be *transcendental* over S if α is not algebraic.

Further:

Definition: If $\forall \alpha \in R$, α is algebraic over S, then R is called an *algebraic extension* of S.

Definition: If $\forall \alpha \in R$, α is integral over S, then R is called an *integral extension* of S.

Transcendence Degree

Let k be a field. Let K be an extension field of k.

Definition: Let S be a subset of K. Let $T = k[X_1, ..., X_n]$ be a polynomial ring in variables $X_1, ..., X_n$. We say S is *algebraically independent* over k when for every finite sequence of distinct elements $x_1, \ldots, x_n \in S$, If $f \in T$, and $f(x_1, \ldots, x_n) = 0$, then f is the zero polynomial.

Definition: Let S be a subset of K. If S is algebraically independent over k, and if the cardinality of S is the greatest cardinality of all algebraically independent subsets of K over k, then the cardinality of S is the *transcendence degree* of K over k.[La]

Laurent Series

Let k be a field with characteristic 0. A *Laurent series* is written as a power series which may include terms of negative degree. A Laurent series with a placeholder x about a given point $a \in k$ (its *center*) is defined as:

$$\sum_{j}^{\infty} a_j (x-a)^j$$

where $j \in \mathbf{Z}$, $a_j \in k$.

Notation: $k((x))$ denotes the ring of Laurent series about 0 in the variable x.

Notation: $k[[x]]$ denotes the ring of power series about 0 in the variable x.

3 Results

Proposition 6 [Ro] Let k be a field of characteristic 0. Let $F,G \in k[[y]][x]$ be monic in x with $\deg_x(F) = n > 0$ and $\deg_x(G) = m > 0$. Let $\rho \in k[[y]]((x^{-1/n}))$ satisfy $F(\rho,y) = x$, where ρ has leading term $x^{1/n}$. Write $G(\rho,y) = \sum_{j=-m}^{\infty} c_j(y)x^{-j/n}$ with $c_j(y) \in k[[y]]$. Then $J(F,G) = 1$ if and only if $c_j(y) \in k$ for all $j : -m \le j \le n-2$ and $c_{n-1}(y) = y/n + c$ for some $c \in k$.

Example 1 Let k be a field of characteristic 0. Let A_0,B_2,B_1,B_0 be algebraically independent over k. Let $R = k[A_0,B_2,B_1,B_0]$. We start our example using the setup from prop 6 of [Ro] by constructing $F,G \in R[x]$ to be monic in x. We let $F = x^2 + A_0$ and let $G = x^3 + B_2x^2 + B_1x + B_0$ so that $\deg_x(F) = 2$ and $\deg_x(G) = 3$. We now want to construct ρ so that it satisfies $F(\rho) = x$, thus $\rho = (x-A_0)^{1/2} = x^{1/2}(1-A_0x^{-1})^{1/2}$. Next we wish to expand this expression for ρ as a Laurent series about 0 in $x^{-1/2}$ using the Binomial Theorem.

$$x^{1/2}(1-x^{-1}A_0)^{1/2} = x^{1/2} \cdot \left(\sum_{j=0}^{\infty} \binom{1/2}{j} (-A_0x^{-1})^j \right)$$

In order to find $G(\rho)$ we will use big-O notation with the series $(1-x^{-1}A_0)^{1/2}$ by truncating it after the third term in the series and adding $O(x^{-3})$.

$$x^{1/2} \left(\sum_{j=0}^{\infty} \binom{1/2}{j} (-A_0x^{-1})^j \right) = x^{1/2} \left(1 + \frac{(-A_0)}{2}x^{-1} - \frac{(-A_0)^2}{8}x^{-2} + O(x^{-3}) \right)$$

If we now construct $G(\rho)$ we get:

$$\begin{aligned}
G(\rho) &= ((x-A_0)^{1/2})^3 + B_2((x-A_0)^{1/2})^2 + B_1(x-A_0)^{1/2} + B_0 \\
&= (x-A_0)(x-A_0)^{1/2} + B_2(x-A_0) + B_1(x-A_0)^{1/2} + B_0 \\
&= (x-A_0+B_1)(x-A_0)^{1/2} + B_2x - A_0B_2 + B_0 \\
&= (x-A_0+B_1)x^{1/2}(1-A_0x^{-1})^{1/2} + B_2x - A_0B_2 + B_0
\end{aligned}$$

$$= \left(x^{3/2} - A_0x^{1/2} + B_1x^{1/2}\right)\left(1 + \frac{(-A_0)}{2}x^{-1} - \frac{(-A_0^2)}{8}(x^{-1})^2 + O(x^{-3})\right) + B_2x + (B_0 - A_0B_2)$$

$$= x^{3/2} + B_2x^{2/2} + \left(\frac{-3A_0}{2} + B_1\right)x^{1/2} + (B_0 - B_2A_0)x^{0/2} + \left(\frac{3A_0^2 - 4B_1A_0}{8}\right)x^{-1/2} + \left(\frac{A_0^3 - B_1A_0^2}{8}\right)x^{-3/2} + O(x^{-3/2})$$

and we can now see the desired coefficients of $G(\rho)$ are:

$$C_1 = B_2,\ C_2 = \left(\frac{-3A_0}{2} + B_1\right),\ C_3 = (B_0 - B_2A_0),\ C_4 = \left(\frac{3A_0^2 - 4B_1A_0}{8}\right)$$

If we construct the ring $S = k[C_1,C_2,C_3,C_4]$ we now wish to show that A_0 is integral over S.

Lemma 1. *Let k be a field with characteristic 0, and let $R = k[a_0, b_0, b_1, b_2]$ and let $S = k[c_1, c_2, c_3, c_4]$. If $c_1 = b_2$, $c_2 = \left(\frac{-3a_0}{2} + b_1\right)$, $c_3 = (b_0 - b_2a_0)$, and $c_4 = \left(\frac{3a_0^2 - 4b_1a_0}{8}\right)$, then a_0 is integral over S.*

PROOF. Assume: $c_1 = b_2$, $c_2 = \left(\frac{-3a_0}{2} + b_1\right)$, $c_3 = (b_0 - b_2a_0)$, and $c_4 = \left(\frac{3a_0^2 - 4b_1a_0}{8}\right)$.

We know $c_2 = \left(\frac{-3a_0}{2} + b_1\right)$ so solving for b_1 we get $b_1 = \frac{3}{2}a_0 + c_2$.

Using this in $c_4 = \left(\frac{3a_0^2 - 4b_1a_0}{8}\right)$ we get by substitution $c_4 = \frac{3a_0^2 - 4\left(\frac{3}{2}a_0 + c_2\right)a_0}{8}$. Simplifying and rewriting this equation in terms of a_0 we have

$$a_0^2 + \frac{4}{3}c_2a_0 + \frac{8}{3}c_4 = 0.$$

Since $1, \frac{4}{3}c_2, \frac{8}{3}c_4 \in S, \exists h(x) = x^2 + \frac{4}{3}c_2x + \frac{8}{3}c_4 \in S[x] - \{0\}$ which is monic and satisfies $h(a_0) = 0$. $\square$

Lemma 2. *Let k be a field of characteristic 0. Let $a_0, b_0, b_1, b_2 \in k[y]$. Let $f = x^2 + a_0$ and let $g = x^3 + b_2x^2 + b_1x + b_0$. If $J(f, g) = 1$, then a_0 is constant in $k[y]$.*

PROOF. Assume $J(f, g) = 1$ and assume for a contradiction a_0 is not constant in $k[y]$. We know from Lemma 1 that $a_0^2 + \frac{4}{3}c_2a_0 + \frac{8}{3}c_4 = 0$. Since a_0 is not constant in $k[y]$ we can say $\deg_y(a_0^2) \geq 2$, and Proposition 6[Ro] tells us $\deg_y(\frac{8}{3}c_4) = 1$, and the strong triangle inequality states $\deg_y(a_0^2 + \frac{8}{3}c_4) = \max\{\deg_y(a_0^2), \deg_y(\frac{8}{3}c_4)\} = \deg_y(a_0^2)$. Also by Proposition 6[Ro] we know c_2 is constant in $k[y]$. Since c_2 is constant in $k[y]$ we know $\deg_y(\frac{4}{3}c_2a_0) \leq \deg_y(a_0^2)$. Since $2(\deg_y(a_0)) = \deg_y(a_0^2)$, we know $\deg_y(\frac{4}{3}c_2a_0) < deg(a_0^2)$. Using the strong triangle inequality we know $\deg_y((a_0^2 + \frac{8}{3}c_4) + \frac{4}{3}c_2a_0) = \max\{\deg_y(a_0^2 + \frac{8}{3}c_4), \deg_y(\frac{4}{3}c_2a_0)\}$ since $\deg_y(a_0^2 + \frac{8}{3}c_4) > \deg_y(\frac{4}{3}c_2a_0)$, thus $\deg_y(a_0^2 + \frac{4}{3}c_2a_0 + \frac{8}{3}c_4) \geq 2$. However, $a_0^2 + \frac{4}{3}c_2a_0 + \frac{8}{3}c_4 = 0$ and $\deg_y(0) = -\infty$. Thus $\deg_y(a_0^2 + \frac{4}{3}c_2a_0 + \frac{8}{3}c_4) = -\infty$, contradiction. $\square$

Proposition 1. *Let k be a field with characteristic 0. Let $f = x^n + \sum_{j=0}^{n-2} a_jx^j$ and $g = x^m + \sum_{j=0}^{m-1} b_jx^j$ with $a_j, b_j \in k[y]$ and $m, n \geq 2$. Let $\mu \in k[y]((x^{-1/n}))$ satisfy $f(\mu, y) = x$, where μ has leading term $x^{1/n}$. Write $g(\mu, y) = \sum_{j=-m}^{\infty} c_j(y)x^{-j/n}$ with $c_j(y) \in k[y]$. Let $A_{n-2}, \ldots, A_0, B_{m-1}, \ldots, B_0$ be algebraically independent variables over k. Let $R = k[A_{n-2}, \ldots, A_0, B_{m-1}, \ldots, B_0]$. Let $F = x^n + \sum_{j=0}^{n-2} A_jx^j$ and $G = x^m + \sum_{j=0}^{m-1} B_jx^j$. Let $\rho \in R((x^{-1/n}))$ satisfy $F(\rho) = x$, where ρ has leading term $x^{1/n}$. Write $G(\rho) = \sum_{j=-m}^{\infty} C_jx^{-j/n}$ with $C_j \in R$. Let $S = k[C_{-m}, \ldots, C_{n-1}]$. Suppose R is integral over S. If $J(f, g) = 1$, then a_0 is constant in $k[y]$.*

PROOF. Assume $J(f, g) = 1$. Also assume for a contradiction a_0 is not constant in k[y]. We start by grading R by assigning $\deg(A_i) = n - i$, $\deg(B_j) = m - j$. Let the degree with respect to this grading on R be denoted as $\deg_R$. Let $\sigma : R \to k[y]$ be the ring homomorphism induced by $\sigma(A_i) \mapsto a_i$, $\sigma(B_j) \mapsto b_j$, and fixing k pointwise. This results in $\sigma(C_j) = c_j$. We know $\exists h \in S[X] - \{0\}$ s.t. $h(A_0) = 0$ and such that h is monic. We can write $h(A_0) = A_0^p + \sum_{j=0}^{p-1} D_jA_0^j$ where $D_j \in k[C_{-m}, \ldots, C_{n-2}][C_{n-1}], p \in \mathbf{N}$, and $\sigma(h(A_0)) = a_0^p + \sum_{j=0}^{p-1} d_ja_0^j$ where $d_j = \sigma(D_j)$. Without loss of generality D_j is homogeneous in R of degree $(p-j)n$, and $\deg_R(A_0^j) = jp$, then $\deg_R(D_jA_0^j) = np$ or $\deg_R(D_jA_0^j) = -\infty$. Define R_i as the homogeneous component of R in degree i. So $D_jA_0^j \in R_{np}$, and $D_j \in R_{n(p-j)}$. Let $q = \deg_{C_{n-1}}(D_j)$. We know $\deg_R(C_{n-1}) = n + m - 1$, and $n(p-j) = \deg_R(D_j) \geq (n+m-1)q$, so $\frac{n(p-j)}{(n+m-1)} \geq q$. Since $\frac{n}{n+m-1} < 1$, $q < p - j$. Proposition 6[Ro] tells us $\deg_y(d_j) \leq q$, then $\deg_y(d_j) < p - j$. We know $\deg_y(d_ja_0^j) = \deg_y(d_j) + \deg_y(a_0^j) < (p-j) + \deg_y(a_0^j) \leq \deg_y(a_0^{p-j}) + \deg_y(a_0^j) = \deg_y(a_o^p)$. Thus $\deg_y(d_ja_0^j) < \deg_y(a_0^p)$. So then the strong triangle inequality tells us $\deg_y(a_0^p + \sum_{j=0}^{p-1} d_ja_0^j) = \deg_y(a_0^p) = p$. However, $a_0^p + \sum_{j=0}^{p-1} d_ja_0^j = \sigma(h(A_0)) = \sigma(0) = 0$. Thus $\deg_y(a_0^p + \sum_{j=0}^{p-1} d_ja_0^j) = -\infty$. Contradiction. $\square$

Example 2 If $R := k[x,y]$ and $S := [x^2-y^2, xy]$, then we will show that R is integral over S.

Let $\gamma = \theta^4 - (x^2-y^2)\theta^2 - (xy)^2$, then $\gamma \in S[\theta] - \{0\}$.

$$\gamma(x) = x^4 - (x^2-y^2)x^2 - (xy)^2 = x^4 - x^4 + y^2x^2 - y^2x^2 = 0$$

From this we can see $\exists \gamma \in S[\theta] - \{0\}$ s.t $\gamma(x) = 0$ and γ is monic. Thus x is integral over S.

Let $\psi = \theta^4 + (x^2-y^2)\theta^2 - (xy)^2$, then $\psi \in S[\theta] - \{0\}$.

$$\psi(y) = y^4 + (x^2-y^2)y^2 - (xy)^2 = y^4 + x^2y^2 - y^4 - x^2y^2 = 0$$

From this we can see $\exists \psi \in S[\theta] - \{0\}$ s.t $\psi(y) = 0$ and ψ is monic. Thus y is integral over S.

Because x, y are both integral over S, then R is integral over S.

Example 3 Let k be a field of characteristic 0. Let A_0, B_3, B_2, B_1, B_0 be algebraically independent over k. Let $F = x^2 + A_0$ and let $G = x^4 + B_3x^3 + B_2x^2 + B_1x + B_0$ so that $\deg_x(F) = 2$ and $\deg_x(G) = 4$. As in Example 1, but expanding to $O(x^{-4})$, we have

$$\rho = x^{1/2}\left(\sum_{j=0}^{\infty}\binom{1/2}{j}(-A_0x^{-1})^j\right) = x^{1/2}\left(1 + \frac{(-A_0)}{2}x^{-1} - \frac{(-A_0)^2}{8}x^{-2} + \frac{(-A_0)^3}{16}x^{-3} + O(x^{-4})\right)$$

If we now construct G(ρ) we get:

$$\begin{aligned}
G(\rho) &= ((x-A_0)^{1/2})^4 + B_3((x-A_0)^{1/2})^3 + B_2((x-A_0)^{1/2})^2 + B_1(x-A_0)^{1/2} + B_0 \\
&= (x-A_0)^2 + B_3(x-A_0)(x-A_0)^{1/2} + B_2(x-A_0) + B_1(x-A_0)^{1/2} + B_0 \\
&= (x-A_0)^2 + (B_3x - A_0B_3 + B_1)(x-A_0)^{1/2} + B_2x - A_0B_2 + B_0 \\
&= (B_3x - A_0B_3 + B_1)\left(x^{1/2} + \frac{(-A_0)}{2}x^{-1/2} - \frac{(-A_0)^2}{8}x^{-3/2} + \frac{(-A_0)^3}{16}x^{-5/2} + O(x^{-7/2})\right) \\
&\quad + x^2 - (2A_2 - B_2)x + A_0^2 + B_0 - A_0B_2 \\
&= x^{4/2} + B_3x^{3/2} + (B_2 - 2A_0)x^{2/2} + \left(B_1 - \frac{3A_0B_3}{2}\right)x^{1/2} + (A_0^2 + B_0 - A_0B_2)x^{0/2} + \left(\frac{3A_0^2B_3}{8} - \frac{A_0B_1}{2}\right)x^{-1/2} \\
&\quad + \left(\frac{A_0^3B_3 - 2B_1A_0^2}{16}\right)x^{-3/2} + \left(\frac{B_3A_0^4 - B_1A_0^3}{16}\right)x^{-5/2} + O(x^{-5/2})
\end{aligned}$$

and we can now see the desired coefficients of $G(\rho)$ are:

$$C_1 = B_3, C_2 = (B_2 - 2A_0), C_3 = \left(B_1 - \frac{3B_3A_0}{2}\right), C_4 = (A_0^2 + B_0 - B_2A_0), C_5 = \left(\frac{3B_3A_0^2}{8} - \frac{B_1A_0}{2}\right)$$

If we construct two rings $R = k[A_0, B_0, B_1, B_2, B_3]$ and $S = k[C_1, C_2, C_3, C_4, C_5]$ we now wish to show that R is not integral over S.

We know $C_1 = B_3$ and $C_3 = \left(B_1 - \frac{3B_3A_0}{2}\right)$ solving for B_3 and B_1 we get $B_3 = C_1$ and $B_1 = C_3 + \frac{3}{2}C_1A_0$. We can then see by substitution that $C_5 = -\frac{3C_1A_0^2 + 4C_3A_0}{8}$. Next we attempt to construct a monic polynomial in A_0.

$$C_5 = -\frac{3C_1A_0^2 + 4C_3A_0}{8}$$

$$-8C_5 = 3C_1A_0^2 + 4C_3A_0$$
$$-3C_1A_0^2 - 4C_3A_0 - 8C_5 = -3C_1A_0^2 - 4C_3A_0 + 3C_1A_0^2 + 4C_3A_0$$
$$3C_1A_0^2 + 4C_3A_0 + 8C_5 = 0$$

So we can see that A_0 is algebraic over S. However, since $3C_1$ is not constant, we can not divide this polynomial by $3C_1$. This is not proof that A_0 is not integral over S. There are an infinite number of polynomials that would need to be checked. So we need to try a new method.

Lemma 3. *Assume that S,R are the rings in Example 3. Let* $\ell = n+m-1$. *Let* $\alpha \in R$. *Suppose* $\exists P \in k^\ell$ *s.t.* $|\delta^{-1}(P)| \nless \infty$, *where* $\xi : k^\ell \to k^{\ell+1}$, $\xi(x_1,\ldots,x_\ell) = (C_1(A_0 = x_1,\ldots,B_{m-1} = x_\ell),\ldots,C_\ell(A_0 = x_1,\ldots,B_{m-1} = x_\ell),\alpha(A_0 = x_1,\ldots,B_{m-1} = x_\ell))$, *and* $\delta : \xi(k^\ell) \to k^\ell$ *where* $(v_1,\ldots,v_\ell,v_{\ell+1}) \mapsto (v_1,\ldots,v_\ell)$. *Then* α *is not integral over S.*

PROOF. Assume for a contradiction α is integral over S. Then $\exists h \in S[x]-0$ s.t. h is monic and $h(\alpha)=0$. Write $h = x^z + D_{z-1}x^{z-1} + \ldots + D_0x^0$, where $D_j \in S$. We can write $P = (p_1,\ldots,p_\ell)$. Now let $\psi = h(C_1 = p_1,\ldots,C_\ell = p_\ell)$. We then have that $\psi \in k[x]$, and $\deg_x(\psi) = z$. This tells us ψ has at most z roots in k. Let $M \in g^{-1}(P)$, write $M = (p_1,\ldots,p_\ell,u)$.There exists $Q \in k^\ell$ such that $\xi(Q) = M$. This tells us $\xi(Q) = \xi(C_i(A_0 = q_1,\ldots,B_{m-1} = q_\ell),\alpha(A_0 = q_1,\ldots,B_{m-1} = q_\ell)) = (p_1,\ldots,p_\ell,u) where Q = (q_1,\ldots,q_\ell)$. So then $u = \alpha(A_0 = q_1,\ldots,B_{m-1} = q_\ell)$. Because of this, $h(x)(A_0 = q_1,\ldots,B_{m-1} = q_\ell) = h(x)(C_1 = p_1,\ldots,C_\ell = p_\ell) = \psi(x)$. We can now see that $0 = h(\alpha) = h(x)(A_0 = q_1,\ldots,B_{m-1} = q_\ell, x = \alpha(A_0 = q_1,\ldots,B_{m-1} = q_\ell)) = h(x)(C_1 = p_1,\ldots,C_\ell = p_\ell, x = u) = \psi(u)$. Since $\psi(u) = 0$ we know u is a root of ψ in k. Thus $\forall N \in \delta^{-1}(P)$, if we write $N = (p_1,\ldots,p_\ell,v)$, then v is a root of ψ in k. Since $|\delta^{-1}(P)| \nless \infty$, then $\deg_x(\psi) \nless \infty$. Contradiction. □

Lemma 4. *Let k be a field of characteristic 0. Let* A_0,B_3,B_2,B_1,B_0 *be algebraically independent variables over k. Define* $\xi : k^5 \to k^6$, $\xi(x_1,\ldots,x_5) = (C_1(A_0 = x_1,\ldots,B_3 = x_5),\ldots,(C_5(A_0 = x_1,\ldots,B_3 = x_5),A_0(A_0 = x_1,\ldots,B_3 = x_5))$, *and* $\delta : \xi(k^5) \to k^5$ *where* $(v_1,\ldots,v_5,v_6) \mapsto (v_1,\ldots,v_5)$. *Let* $R = k[A_{n-2},\ldots,A_0,B_{m-1},\ldots,B_0]$. *Let* $F = x^2 + A_0$ *and* $G = x^4 + B_3x^3 + B_2x^2 + B_1x + B_0$. *Let* $\rho \in R((x^{-1/n}))$ *satisfy* $F(\rho) = x$, *where* ρ *has leading term* $x^{1/n}$. *Write* $G(\rho) = \sum_{j=-m}^{\infty} C_j x^{-j/n}$ *with* $C_j \in R$. *Let* $S = k[C_1,C_2,C_3,C_4,C_5]$. *Then* A_0 *is not integral over S.*

PROOF. Let $P = (0, 0, 0, 0, 0)$. Let $\alpha = A_0$. Let $u \in k$. Write $M = (0,0,0,0,0,u)$. We claim that $\exists\ Q \in k^5$ s.t. $\xi(Q) = M$. Let $H = F$, let $F_1 = x$, and let $G_1 = x^2$. Then let $\tilde{G} = G_1 \circ H = (x^2 + A_0)^2 = x^4 + 2A_0x^2 + A_0^2$, and let $\tilde{F} = F_1 \circ H = x^2 + A_0$. Then $\tilde{G} \circ \tilde{F}^{-1} = (G_1 \circ H) \circ (F_1 \circ H)^{-1} = (G_1 \circ H) \circ (H^{-1} \circ F_1^{-1}) = G_1 \circ F_1^{-1} = (x)^2 = x^2$. Let $Q = (u,u^2,0,2u,0)$. Since $\tilde{G} \circ \tilde{F}^{-1} = x^2$ and $\xi(Q) =$ (the first five coefficients after x^2 of $\tilde{G} \circ \tilde{F}^{-1}(A_0 = u)$, $A_0(A_0 = u, B_0 = u^2, B_1 = 0, B_2 = 2u, B_3 = 0)) =$ (the first five coefficients after x^2 of $x^2, u) = (0,0,0,0,0,u) = M$. So since u is an arbitrary element of k, and since k is a field with characteristic 0, then u can take an infinite number of values. Thus $|\delta^{-1}(P)| \nless \infty$. So by Lemma 3 we know A_0 is not integral over S. □

4 Bibliography

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