A CONJUGATE DECOMPOSITION OF THE EUCLIDEAN SPACE

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ABSTRACT

Given a closed convex cone K in the n-dimensional real Euclidean space R^n and an $n\times n$ real matrix A which is positive definite on K, we show that each vector in R^n can be decomposed into a component which lies in K and another which lies in the conjugate cone induced by A and such that the two vectors are conjugate to each other with respect to $A + A^T$. As a consequence of this decomposition we establish the following characterization of positive definite matrices: An $n\times n$ real matrix A is positive definite if and only if it is positive definite on some closed convex cone K in R^n and $(A+A^T)^{-1}$ exists and is positive semidefinte on the polar cone K^0. If K is a subspace of R^n then K^0 is its orthogonal complement $K^\perp$. Other applications include local duality results for nonlinear programs and other characterizations of positive definite and semidefinite matrices.

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Let A be an $n \times n$ real matrix and K be a closed convex cone in the n-dimensional real Euclidean space R^n. A vector a in R^n is said to have a conjugate decomposition with respect to A and K if there exists an x in K and a y in the conjugate cone $K^A := \{y | y^T(A+A^T)x \leqq 0, \forall x \in K\}$ such that

$$a = x + y \quad \text{and} \quad x^T(A+A^T)y = 0.$$

When K is a subspace and A is the identity matrix such a decomposition becomes the classical orthogonal decomposition of a vector into its projections onto the subspace K and its orthogonal complement $K^{\perp}$. It is well known that, in this case, such a decomposition exists and is unique for any given vector a. This result was generalized by Moreau [5] to the case where K is any closed cone in a Hilbert space with any Hilbertian norm. Thus, if A is positive definite on the entire space, it defines a norm $\|x\|_A^2 := x^TAx$, and our decomposition result, Theorem 1, follows directly from Moreau's theorem. The main point of this paper is to extend Moreau's result to the case where A is not positive definite on the entire space but merely on the closed convex cone K, that is $x^TAx > 0$ whenever $0 \neq x \in K$. Although the results of this paper are extendable to a Hilbert space, they are presented here only for a real Euclidean space. We begin with our first principal result.

<u>Theorem 1</u> Let A be an $n \times n$ real matrix and K be any closed convex cone in R^n. If A is positive definite on K then any vector in R^n has a conjugate decomposition with respect to A and K. Moreover, if A is positive definite on the linear hull of K then the decomposition is unique.

Proof (Existence) Let a be a given fixed vector in R^n, and let

$$S := \{x \mid x \in K \text{ and } \|x\| \leqq \frac{\|a^T(A+A^T)\|}{\alpha}\}$$

where $\|\cdot\|$ denotes the Euclidean norm and

$$\alpha := \min \{x^T Ax \,\big|\, \|x\| = 1,\ x \in K\} > 0.$$

Let $f(x) := (x-a)^T A(x-a)$ and consider the following problems

(Q) $\min \{f(x) \mid x \in K\},$

(Q′) $\min \{f(x) \mid x \in S\}.$

Any solution of (Q′) also solves (Q) because for any $x \in K \backslash S$,

$$\begin{aligned} f(x) &= x^T Ax - a^T(A+A^T)x + a^T Aa \\ &\geqq (\alpha\|x\| - \|a^T(A+A^T)\|)\,\|x\| + a^T Aa \\ &> f(0). \end{aligned}$$

It follows from the compactness of S that (Q) has a solution $\bar{x}$, say. Then by the minimum principle [4, Theorem 9.3.3], we have that

$$(x-\bar{x})^T(A+A^T)(\bar{x}-a) \geqq 0 \qquad \forall x \in K.$$

By letting $x = 2\bar{x}$ and $x = 0$ and letting $\bar{y} = a - \bar{x}$, we have that

$$a = \bar{x} + \bar{y},\ \bar{x} \in K,\ \bar{y} \in K^A \text{ and } \bar{x}^T(A+A^T)\bar{y} = 0,$$

which is a conjugate decomposition of a with respect to A and K.

(Uniqueness) Let $a = \hat{x} + \hat{y} = \bar{x} + \bar{y}$ be two conjugate decompositions of a. Then it follows from $\bar{x} - \hat{x} = \hat{y} - \bar{y}$ that

$$(\bar{x}-\hat{x})^T A(\bar{x}-\hat{x}) = \frac{1}{2}(\bar{x}-\hat{x})^T(A+A^T)(\hat{y}-\bar{y})$$
$$= \frac{1}{2}\bar{x}^T(A+A^T)\hat{y} + \frac{1}{2}\hat{x}^T(A+A^T)\bar{y}$$
$$\leqq 0$$

Since A is positive definite on the linear hull of K, the last inequality can hold only when $\bar{x} = \hat{x}$. Hence, the decomposition is unique. □

An important consequence of Theorem 1 is the following characterization of positive definite matrices.

Theorem 2 Let A be an $n \times n$ real matrix and let K be a closed convex cone in R^n. A is positive definite if and only if A is positive definite on K and $(A+A^T)^{-1}$ exists and is positive semi-definite on the polar cone $K^0 := \{y | y^T x \leqq 0, \forall x \in K\}$.

Proof The "only if" is trivially true. Let a be any given vector in R^n, then by Theorem 1, there exists a conjugate decomposition $a = \bar{x} + \bar{y}$ with $\bar{x} \in K$, $\bar{y} \in K^A$ and $\bar{x}^T(A+A^T)\bar{y} = 0$. Let $\bar{z} = (A+A^T)\bar{y}$ then $\bar{z} \in K^0$. Thus

$$a^T A a = (\bar{x}+\bar{y})^T A(\bar{x}+\bar{y})$$
$$= \bar{x}^T A\bar{x} + \frac{1}{2}\bar{y}^T(A+A^T)\bar{y}$$
$$= \bar{x}^T A\bar{x} + \frac{1}{2}\bar{z}^T(A+A^T)^{-1}\bar{z} \geqq 0 .$$

Hence, A is positive semidefinite and so is $A + A^T$. Since $A + A^T$ is nonsingular, $A + A^T$ is in fact positive definite and so is A. □

A direct consequence of Theorem 2 is the following.

Corollary 3 Let A be an $n \times n$ real matrix and let K be a closed convex cone in R^n such that $-K^0 \subset K$. A is positive definite if and only if A is positive definite on K and $(A+A^T)^{-1}$ exists and is positive semidefinite on K.

If we let $K = \{x | Bx \leqq 0\}$ in Corollary 3 where B is some $m \times n$ real matrix, then $-K^0 = \{y | x^T y \geqq 0, \forall x \in K\} = \{y | y = -B^T u, u \geqq 0\}$. Hence $-K^0 \subset K$ if and only if $BB^T u \geqq 0$ for all $u \geqq 0$ or equivalently if $BB^T \geqq 0$. Consequently we have the following.

Corollary 4 Let A be $n \times n$ real matrix and let B be an $m \times n$ real matrix such that $BB^T \geqq 0$. A is positive definite if and only if A is positive definite on $K = \{x | Bx \leqq 0\}$ and $(A+A^T)^{-1}$ exists and is positive semidefinite on K.

By letting B be the negative of the identity matrix in Corollary 4 we obtain the following interesting characterization of positive definite matrices in terms of strictly copositive and copositive matrices.

Corollary 5 A necessary and sufficient condition for an $n \times n$ real matrix A to be positive definite is that A be strictly copositive (that is $x^T A x > 0$ for $0 \neq x \geqq 0$) and $A + A^T$ has a copositive inverse (that is $x^T (A+A^T)^{-1} x \geqq 0$ for all $x \geqq 0$).

By letting K in Theorem 2 be a subpace of R^n, we get the following result obtained in [1] by a different technique which does not extend to cones.

Corollary 6 Let A be an $n \times n$ real symmetric matrix and K be a subspace of R^n. A is positive definite if and only if A is positive definite on K and A^{-1} exists and is positive semidefinite on the orthogonal complement $K^{\perp}$ of K.

Applications of Corollary 6 and Theorem 2 to local duality results of nonlinear programming are given in [1,3]. Additional results pertaining to conjugate decomposition with respect to positive semidefinite matrices are given in [2]. Other possible applications are to the theory of penalty functions and augmented Lagrangians [6].

References

1. O. Fujiwara, S.-P. Han and O. L. Mangasarian: "Local duality of nonlinear programs", Mathematics Research Center, University of Wisconsin, Madison, Technical Summary Report 2329, February 1982, to appear, SIAM Journal on Control and Optimization.

2. S.-P. Han and O. L. Mangasarian: "Conjugate cone characterization of positive definite and semidefinite matrices", Computer Sciences Department, University of Wisconsin, Madison, Technical Report #471, March 1982, to appear, Linear Algebra and Its Applications.

3. S.-P. Han and O. L. Mangasarian: "Characterization of positive definite and semidefinite matrices via quadratic programming duality", Computer Sciences Department, University of Wisconsin, Madison, Technical Report #473, June 1982, to appear, SIAM Journal on Algebraic and Discrete Methods.

4. O. L. Mangasarian: "Nonlinear programming", McGraw-Hill, New York, 1969.

5. J. J. Moreau: "Décomposition orthogonale d'un espace hilbertien selon deux cône mutuellement polaires", C. R. Acad. Sci. Paris 255, 1962, 238-240.

6. A. P. Wierzbicki and S. Kurcyusz: "Projection on a cone, penalty functionals and duality theory for problems with inequality constraints in Hilbert space", SIAM Journal on Control and Optimization 15, 1977, 25-56.