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A CO-FACTOR IDENTITY FOR COMPOUND MATRICES

by

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ABSTRACT

This note defines a linear ordering of the set Q_k^m of all k-member subsets $J = \{j_1, j_2, \ldots, j_k\}$ of $\{1, 2, \ldots, m\}$, with $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant m$, by $I < J$ iff $(\exists r)\ i_r < j_r$ and $(\forall s > r)\ i_s = j_s$, and shows (in *Lemma 1*) that this corresponds to the ordinal function $\lambda_k^m(J) = 1 + \binom{j_1-1}{1} + \binom{j_2-1}{2} + \ldots + \binom{j_k-1}{k}$. If $\mathbf{A}$ is an $(m \times m)$ matrix with elements $(\mathbf{A})_{ij} = \alpha_{ij}$, this identifies the k-th compound matrix $\mathbf{A}^{(k)}$ with the k-rowed minor $\alpha_{IJ}^{(k)}$ of $D = \det \mathbf{A}$ as $(\mathbf{A}^{(k)})_{uv}$, with $u = \lambda_k^m(I)$ and $v = \lambda_k^m(J)$. The main *Theorem* then shows that, if $\Lambda(\alpha_{IJ}^{(k)})$ is the co-factor of $\alpha_{IJ}^{(k)}$ in D, and $\Lambda^{(k)}(\alpha_{IJ}^{(k)})$ is the co-factor of the same $\alpha_{IJ}^{(k)}$ in $D^{(k)} = \det \mathbf{A}^{(k)}$, then $\Lambda^{(k)}(\alpha_{IJ}^{(k)})/\Lambda(\alpha_{IJ}^{(k)}) = D^{c-1}$, where $c = \binom{m-1}{k-1}$. From this are derived three corollaries and a further lemma; and, as a fourth corollary, the well-known Jacobi Identity is obtained. In the process, the main properties of determinants, compounds, adjugates, reciprocals, and co-factors are summarized.

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Let $\mathbf{M}_{mn}$ denote the set of all $(m \times n)$ [m rows, n columns] matrices with real elements; if $\mathbf{X} \in \mathbf{M}_{mn}$, let $(\mathbf{X})_{ij} = \xi_{ij}$ denote the element in row i $(1 \leqslant i \leqslant m)$ and column j $(1 \leqslant j \leqslant n)$; and let $\mathbf{I} = \mathbf{I}_m \in \mathbf{M}_{mm}$ denote the m-rowed *unit* (or *identity*) matrix, with elements $(\mathbf{I})_{ij} = \delta_{ij}$ [δ_{ij} being the Kronecker function: 0 if $i \neq j$, 1 if $i = j$.] If $\mathbf{Y} \in \mathbf{M}_{np}$, with $(\mathbf{Y})_{jh} = \eta_{jh}$, then the *matrix product* $\mathbf{X}\,\mathbf{Y} = \mathbf{Z} \in \mathbf{M}_{mp}$, where $(\mathbf{Z})_{ih} = \zeta_{ih} = \Sigma_{j=1}^{n} \xi_{ij}\, \eta_{jh}$. Similarly, if $\mathbf{A} \in \mathbf{M}_{mm}$ and $\mathbf{B} \in \mathbf{M}_{mn}$, with $(\mathbf{A})_{ij} = \alpha_{ij}$ and $(\mathbf{B})_{ij} = \beta_{ij}$, we may define the equation $\mathbf{A}\,\mathbf{X} = \mathbf{B}$, with $\mathbf{A}$ and $\mathbf{B}$ given and $\mathbf{X}$ unknown. The determinant of the *square matrix* $\mathbf{A}$ is defined as

$$D = \det \mathbf{A} = |\alpha_{i1}\ \alpha_{i2}\ \cdots\ \alpha_{im}| = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2m} \\ & \cdots & \cdots & \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mm} \end{vmatrix}$$

$$= \Sigma_{\rho \in P_m}\ \varepsilon_\rho\ \alpha_{1\rho(1)}\ \alpha_{2\rho(2)}\ \cdots\ \alpha_{m\rho(m)}, \tag{1}$$

where P_m denotes the set of all *permutations* of $N = N_m = \{1, 2, \ldots, m\}$ and ε_ρ is the *parity index* of the permutation ρ (taking values ± 1: $+1$ if ρ may be represented by an even number of interchanges, -1 otherwise.) Then, if $D \neq 0$, the Leibnitz-Cramer rule[1] (generalized in the obvious way to the matrix equation $\mathbf{A}\,\mathbf{X} = \mathbf{B}$) tells us that

$$\xi_{ij} = D_{ij}/D, \tag{2}$$

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[1] See reference [1] p. 134.

where D_{ij} denotes the determinant obtained by replacing the i-th column of the determinant D of **A** by the j-th column of the matrix **B**.

A determinant may be expanded by any row or column:

$$\left.\begin{aligned} D &= \sum_{j=1}^{m} (-1)^{i+j} \alpha_{ij}\, \Omega(\alpha_{ij}) = \sum_{j=1}^{m} \alpha_{ij}\, \Lambda(\alpha_{ij}) \\ \text{and} \quad D &= \sum_{i=1}^{m} (-1)^{i+j} \alpha_{ij}\, \Omega(\alpha_{ij}) = \sum_{i=1}^{m} \alpha_{ij}\, \Lambda(\alpha_{ij}); \end{aligned}\right\} \tag{3}$$

where $\Omega(\alpha_{ij})$ denotes the *complementary minor* to α_{ij} in D, so that, if ^{c} denotes the complementation of a set and we write $\alpha_{IJ}^{(k)}$ for the minor of **A** consisting of the k rows indexed in I and the k columns indexed in J, then

$$\Omega(\alpha_{ij}) = \alpha_{\{i\}^c\{j\}^c}^{(m-1)}; \tag{4}$$

and where $\Lambda(\alpha_{ij})$ denotes the *co-factor* of α_{ij} in D, so that

$$\Lambda(\alpha_{ij}) = (-1)^{i+j}\, \Omega(\alpha_{ij}). \tag{5}$$

As is well-known[2], (3) may be extended to state that

$$\sum_{j=1}^{m} \alpha_{hj}\, \Lambda(\alpha_{ij}) = \delta_{hi}\, D \quad \text{and} \quad \sum_{i=1}^{m} \alpha_{ih}\, \Lambda(\alpha_{ij}) = \delta_{hj}\, D. \tag{6}$$

The Laplace Expansion Theorem[3] states that, if we select any k rows $I = \{i_1, i_2, \ldots, i_k\}$ or k columns $J = \{j_1, j_2, \ldots, j_k\}$ from N_m and write

$$\sigma_k^m(J) = \sum_{s=1}^{k} j_s, \tag{7}$$

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2 See [1] p. 20.

3 See [1] p. 21, or [2] p. 14.

then

$$\left.\begin{aligned} D &= \sum_{J\in Q_k^m} (-1)^{\sigma_k^m(I)+\sigma_k^m(J)} \alpha_{IJ}^{(k)}\ \Omega(\alpha_{IJ}^{(k)}) = \sum_{J\in Q_k^m} \alpha_{IJ}^{(k)}\ \Lambda(\alpha_{IJ}^{(k)}) \\ \text{and}\quad D &= \sum_{I\in Q_k^m} (-1)^{\sigma_k^m(I)+\sigma_k^m(J)} \alpha_{IJ}^{(k)}\ \Omega(\alpha_{IJ}^{(k)}) = \sum_{I\in Q_k^m} \alpha_{IJ}^{(k)}\ \Lambda(\alpha_{IJ}^{(k)}); \end{aligned}\right\} \quad (8)$$

where $\Omega(\alpha_{IJ}^{(k)})$ denotes the complementary minor to $\alpha_{IJ}^{(k)}$ in D, so that

$$\Omega(\alpha_{IJ}^{(k)}) = \alpha_{I^c J^c}^{(m-k)}; \quad (9)$$

and where $\Lambda(\alpha_{IJ}^{(k)})$ denotes the co-factor of $\alpha_{IJ}^{(k)}$ in D, so that

$$\Lambda(\alpha_{IJ}^{(k)}) = (-1)^{\sigma_k^m(I)+\sigma_k^m(J)}\ \Omega(\alpha_{IJ}^{(k)}). \quad (10)$$

By an argument analogous to that yielding (6) from (3) [based on the fact that a determinant with two rows or two columns identical vanishes], we can extend (8) to yield that

$$\sum_{J\in Q} \alpha_{HJ}^{(k)}\ \Lambda(\alpha_{IJ}^{(k)}) = \delta_{HI}\, D \quad\text{and}\quad \sum_{I\in Q} \alpha_{IH}^{(k)}\ \Lambda(\alpha_{IJ}^{(k)}) = \delta_{HJ}\, D, \quad (11)$$

where

$$\delta_{IJ} = \delta_{i_1 j_1}\ \delta_{i_2 j_2}\ \cdots\ \delta_{i_k j_k}. \quad (12)$$

The *adjugate*[4] (or *adjoint*) of the matrix $\mathbf{A}$ is defined as the matrix $\mathbf{A}^{\mathbf{A}}$ with elements

$$(\mathbf{A}^{\mathbf{A}})_{ji} = \Lambda(\alpha_{ij}); \quad (13)$$

and, if $D \neq 0$, the *reciprocal*[5] (or *inverse*) matrix $\mathbf{A}^{-1}$ of $\mathbf{A}$ is defined by

$$\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}; \quad (14)$$

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[4] See [1] pp. 24, 88, or [2] p. 13.

[5] See [1] p. 91, or [2] p. 3.

so that, by (8) and (13), since the reciprocal is unique,

$$\mathbf{A}^{-1} = D^{-1}\,\mathbf{A}^{\mathbf{A}}. \tag{15}$$

Let $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, j_2, \ldots, j_k\}$ be sets of k distinct indices selected from N_m; to be specific, let these be in ascending order:

$$1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant m \quad \text{and} \quad 1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant m. \tag{16}$$

Let us write $I = J$ *iff* [if and only if] $(\forall s \in N_m)\ i_s = j_s$; and $I < J$ *iff* $(\exists r \in N_m)\ i_r < j_r$ and $(\forall s > r)\ i_s = j_s$. It is easily verified that this is a total ordering of the set $Q = Q_k^m$ of all $\binom{m}{k}$ selections of k distinct indices from N_m, and that this ordering coincides both with the *lexical* order of the "words" $j_k j_{k-1} \ldots j_2 j_1$ and with the ascending order of the numbers $\ell_k^m(J) = \ell_k^m(j_1, j_2, \ldots, j_k) = j_1 m + j_2 m^2 + \ldots + j_k m^k$.

Lemma 1. *The function λ_k^m defined by*

$$\lambda_k^m(J) = \lambda_k^m(j_1, j_2, \ldots, j_k) = 1 + \binom{j_1-1}{1} + \binom{j_2-1}{2} + \ldots + \binom{j_k-1}{k} \tag{17}$$

is a bijection from the set Q onto the set N_q with $q = \binom{m}{k}$; and the ordering of Q defined by $\lambda_k^m(I) < \lambda_k^m(J)$ corresponds to the ordering $I < J$ defined above.

⟦*Proof.* $I < J$ *iff* one of the (clearly) mutually exclusive conditions, $i_r < j_r$ and $(\forall s > r)\ i_s = j_s$ holds, for some $1 \leqslant r \leqslant k$. The number of sets I such that $I < J$ is therefore equal to the sum of the number of ways of choosing $1 \leqslant i_1 < i_2 < \ldots < i_r < j_r$, and this is $\binom{j_r-1}{r}$. Thus, $\lambda_k^m(J)$ is the ordinal number of J in Q_k^m. ⟧

With

$$q = q(m, k) = \binom{m}{k} \quad \text{and} \quad c = c(m, k) = \binom{m-1}{k-1}, \tag{18}$$

the $(q \times q)$ matrix $\mathbf{A}^{(k)}$ whose elements are the k-rowed minors of $\mathbf{A}$

$$(\mathbf{A}^{(k)})_{IJ} = \alpha_{IJ}^{(k)} = \begin{vmatrix} \alpha_{i_1 j_1} & \alpha_{i_1 j_2} & \cdots & \alpha_{i_1 j_k} \\ \alpha_{i_2 j_1} & \alpha_{i_2 j_2} & \cdots & \alpha_{i_2 j_k} \\ \cdots & \cdots & & \cdots \\ \alpha_{i_k j_1} & \alpha_{i_k j_2} & \cdots & \alpha_{i_k j_k} \end{vmatrix}, \tag{19}$$

with the compound indices I and J ordered by the function λ_k^m, as was established in Lemma 1, is called the *k-th compound matrix*[6] *of* $\mathbf{A}$. We note that

$$\mathbf{A}^{(1)} = \mathbf{A} \quad \text{and} \quad \mathbf{A}^{(m)} = \det \mathbf{A} = D; \tag{20}$$

that the k-th compound of the unit matrix is a unit matrix

$$\mathbf{I}_m^{(k)} = \mathbf{I}_q \tag{21}$$

⟦By (19), $(\mathbf{I}_m^{(k)})_{IJ}$ has as its (r, s)-entry $\delta_{i_r j_s}$; so that, if $I = J$, this determinant is $\det \mathbf{I}_k = 1$, while, if $I \neq J$, it has at least one null row, and so vanishes; or, in other words, $(\mathbf{I}_m^{(k)})_{IJ} = \delta_{IJ}$, yielding (21)⟧; and that

$$(\gamma\mathbf{A})^{(k)} = \gamma^k \mathbf{A}^{(k)} \tag{22}$$

⟦$(\gamma\mathbf{A})_{ij} = \gamma\, \alpha_{ij}$, and $\det(\gamma\mathbf{A}) = \gamma^m \det \mathbf{A}$; so, by (19), $((\gamma\mathbf{A})^{(k)})_{IJ} = \gamma^k \alpha_{IJ}^{(k)}$, and (22) follows.⟧ The Binet-Cauchy Theorem[7] asserts that

$$(\mathbf{X}\,\mathbf{Y})^{(k)} = \mathbf{X}^{(k)}\, \mathbf{Y}^{(k)}; \tag{23}$$

whence, if $\mathbf{A}\,\mathbf{X} = \mathbf{B}$, then

$$\mathbf{A}^{(k)}\, \mathbf{X}^{(k)} = \mathbf{B}^{(k)}. \tag{24}$$

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6 See [2] p. 16.

7 See [2] p. 14.

Let us write $D^{(k)} = \det \mathbf{A}^{(k)}$ and, by analogy with D_{ij}, let $D_{IJ}^{(k)}$ denote the determinant obtained by replacing the $\lambda_k^m(I)$-th column of the determinant $D^{(k)}$ of $\mathbf{A}^{(k)}$ by the $\lambda_k^m(J)$-th column of the matrix $\mathbf{B}^{(k)}$. Then, by the Leibnitz-Cramer Rule[1] applied directly to (24), we get

$$\xi_{IJ}^{(k)} = D_{IJ}^{(k)}/D^{(k)}, \tag{25}$$

if $D^{(k)} \neq 0$, just like (2). In addition, the Sylvester-Franke Theorem[8] asserts that

$$D^{(k)} = D^{c}, \tag{26}$$

so that (25) applies if $D \neq 0$.

By (14), (21), and (23), we see that *the compound of the reciprocal matrix is the reciprocal of the compound matrix*:

$$\mathbf{A}^{-1(k)} = \mathbf{A}^{(k)-1}. \tag{27}$$

Finally, by applying (5) and (6) to the compound matrix $\mathbf{A}^{(k)}$, we obtain that

$$\Lambda^{(k)}(\alpha_{IJ}^{(k)}) = (-1)^{\lambda_k^m(I)+\lambda_k^m(J)}\ \Omega^{(k)}(\alpha_{IJ}^{(k)}), \tag{28}$$

where $\Omega^{(k)}(\alpha_{IJ}^{(k)})$ denotes the minor of $D^{(k)}$ complementary to the element $\alpha_{IJ}^{(k)}$, and $\Lambda(\alpha_{IJ}^{(k)})$ denotes the co-factor of $\alpha_{IJ}^{(k)}$ in $D^{(k)}$; and

$$\sum_{J\in Q} \alpha_{HJ}^{(k)}\ \Lambda(\alpha_{IJ}^{(k)}) = \delta_{HI}\ D^{c} \quad \text{and} \quad \sum_{I\in Q} \alpha_{IH}^{(k)}\ \Lambda(\alpha_{IJ}^{(k)}) = \delta_{HJ}\ D^{c}, \tag{29}$$

by (26). We are now ready to prove our main result:

Theorem. *The co-factors of* $\alpha_{IJ}^{(k)}$ *in* D *and in* $D^{(k)}$ *are related by the identity*

$$\frac{\Lambda^{(k)}(\alpha_{IJ}^{(k)})}{\Lambda(\alpha_{IJ}^{(k)})} = D^{c-1}. \tag{30}$$

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[8] See [2] p. 17.

⟦*Proof*. Use the second equation (the sum by columns) of (11) and the first equation (the sum by rows) of (29) to yield that

$$D\ \Lambda^{(k)}(\alpha_{IJ}^{(k)}) = \Sigma_K\ \Lambda^{(k)}(\alpha_{IK}^{(k)})\ \delta_{KJ}\ D = \Sigma_K\ \Sigma_H\ \Lambda^{(k)}(\alpha_{IK}^{(k)})\ \alpha_{HK}^{(k)}\ \Lambda(\alpha_{HJ}^{(k)})$$
$$= \Sigma_H\ \delta_{HI}\ D^c\ \Lambda(\alpha_{HJ}^{(k)}) = D^c\ \Lambda(\alpha_{IJ}^{(k)}),$$

and (30) follows.⟧

Corollary 1. *The minors complementary to* $\alpha_{IJ}^{(k)}$ *in* D *and in* $D^{(k)}$ *are related by the identity*

$$\frac{\Omega^{(k)}(\alpha_{IJ}^{(k)})}{\Omega(\alpha_{IJ}^{(k)})} = D^{c-1}\ (-1)^{\lambda_k^m(I)+\lambda_k^m(J)+\sigma_k^m(I)+\sigma_k^m(J)}. \tag{31}$$

⟦*Proof*. This follows immediately from (10), (28), and (30).⟧

Corollary 2. *The adjugate of* $\mathbf{A}^{(k)}$ *is given by*

$$(\mathbf{A}^{(k)\mathbf{A}})_{JI} = D^{c-1}\ \Lambda(\alpha_{IJ}^{(k)}). \tag{32}$$

⟦*Proof*. By the definition (13), applied to the matrix $\mathbf{A}^{(k)}$, $(\mathbf{A}^{(k)\mathbf{A}})_{JI} = \Lambda^{(k)}(\alpha_{IJ}^{(k)})$. Now (32) follows by application of (30).⟧

Corollary 3. *If* $D \neq 0$, *the reciprocal of* $\mathbf{A}^{(k)}$ *is given by*

$$\mathbf{A}^{(k)-1} = D^{-c}\ \mathbf{A}^{(k)\mathbf{A}}. \tag{33}$$

⟦*Proof*. By (15) and (26), $\mathbf{A}^{(k)-1} = D^{(k)-1}\ \mathbf{A}^{(k)\mathbf{A}}$ and (33) follows.⟧

By (27), Corollary 3 gives us all we need to know about compounds of reciprocals of matrices. For adjugates, we have:

Lemma 2. *The adjugate of the compound is related to the compound of the adjugate by the identity*

$$D^k\ \mathbf{A}^{(k)\mathbf{A}} = D^c\ \mathbf{A}^{\mathbf{A}(k)}. \tag{34}$$

⟦*Proof.* If $D = 0$, (34) is trivially true. So suppose that $D \neq 0$. By Corollary 3, $D^k\,\mathbf{A}^{(k)\mathrm{A}} = D^{k+c}\,\mathbf{A}^{(k)-1}$; and, by (15) and (22), $D^c\,\mathbf{A}^{\mathrm{A}(k)} = D^c\,(D\mathbf{A}^{-1})^{(k)} = D^{c+k}\,\mathbf{A}^{-1(k)}$: by (27), (34) follows.⟧

Another consequence of (27), (33), and (34) is that, if $D \neq 0$,

$$\mathbf{A}^{(k)-1} = \mathbf{A}^{-1(k)} = D^{-k}\,\mathbf{A}^{\mathrm{A}(k)}. \tag{35}$$

Corollary 4 (Jacobi's Identity)[9]. *If* $\alpha_{IJ}^{(k)}$ *is a k-rowed minor of a matrix* $\mathbf{A}$, $(\mathbf{A}^{\mathrm{A}})_{JI}^{(k)}$ *is the corresponding minor of the adjugate matrix* $\mathbf{A}^{\mathrm{A}}$, *and* $\Lambda(\alpha_{IJ}^{(k)})$ *is the co-factor of* $\alpha_{IJ}^{(k)}$; *then*

$$(\mathbf{A}^{\mathrm{A}})_{JI}^{(k)} = D^{k-1}\,\Lambda(\alpha_{IJ}^{(k)}). \tag{36}$$

⟦*Proof.* By (32), with (34), $(\mathbf{A}^{\mathrm{A}})_{JI}^{(k)} = D^{k-c}\,(\mathbf{A}^{(k)\mathrm{A}})_{JI} = D^{k-c}\,D^{c-1}\,\Lambda(\alpha_{IJ}^{(k)}) = D^{k-1}\,\Lambda(\alpha_{IJ}^{(k)})$.⟧

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[1] L. MIRSKY. *An Introduction to Linear Algebra.* Clarendon Press, Oxford, 1955: corrected 1972.

[2] M. MARCUS and H. MINC. *A Survey of Matrix Theory and Matrix Inequalities.* Allyn and Bacon, Boston, 1964.

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[9] See [1] p. 25. Note that the *adjugate determinant* referred to by Mirsky is defined as the determinant of the *transpose* of the adjugate matrix (which does not matter, since det $\mathbf{A}$ = det $\mathbf{A}^{\mathrm{T}}$), accounting for the transposition of compound indices "JI" (not "IJ") on the left-hand side of (36).